BASIC EXAMPLES OF MEAN CURVATURE 1 SURFACES IN $\boldsymbol{H}^3$

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Abstract. We describe the theory of complete mean curvature one surfaces in H^3 and give a detailed description of some basic examples. This note is intended to complement the three EG-Models 2001.01.048 and 2001.01.049 and 2001.01.050 by the authors on this subject.

1. Introduction

There is a wide body of knowledge about minimal surfaces in Euclidean 3-space $\boldsymbol{R}^3$, and there is a canonical local isometric correspondence (sometimes called the Lawson correspondence, canonical correspondence, or cousin correspondence) between minimal surfaces in $\boldsymbol{R}^3$ and CMC-1 (constant mean curvature one) surfaces in hyperbolic 3-space H^3 (the complete simply-connected 3-manifold of constant sectional curvature -1). This has naturally led to the recent interest in and development of CMC-1 surfaces in H^3 in the last decade. There are now many known examples, and here we give a detailed description of some of the most basic ones.

By this canonical local isometric correspondence, minimal immersions in $\boldsymbol{R}^3$ are locally equivalent to CMC-1 immersions in H^3. But there are interesting differences between these two types of immersions on the global level. There are period problems for an immersion of a non-simply-connected domain, which might be solved for the immersion into one 3-space, but not the other. Solvability of the period problems is usually more likely in the H^3 case, leading to a wider variety of surfaces there. For example, a genus 0 surface with finite total curvature and two embedded ends exists as a minimal surface in $\boldsymbol{R}^3$ only if it is a surface of revolution, but it may exist as a CMC-1 surface in H^3 without being a surface of revolution (see Example 4.3).

2. General comments on total absolute curvature

The total absolute curvature of a minimal surface in $\boldsymbol{R}^3$ is equal to the area of the image (counted with multiplicity) of the Gauss map of the surface. Furthermore, as the Gauss map of a complete conformally parametrized finite-total-curvature minimal surface is meromorphic at each end, the area of the Gauss image must be an integer multiple of 4π.

However, unlike minimal surfaces in $\boldsymbol{R}^3$, we have a choice of two different Gauss maps for CMC-1 surfaces in H^3: the *hyperbolic Gauss map* G and the *secondary Gauss map* g. The true total absolute curvature is the area of the image of g, but since g might not be single-valued on the surface, the total curvature might not be an integer multiple of 4π. The Osserman inequality does not hold for the true total absolute curvature. The weaker Cohn-Vossen inequality is the best general lower bound for true absolute total curvature (with equality never holding [UY1]).

The area of the image of G, which we call the *dual* total absolute curvature, is the true total curvature of the dual CMC-1 surface (which we define in Section 3) in H^3. G is single-valued on the surface, and so the dual total absolute curvature is always an integer multiple of 4π, like the case of minimal surfaces in $\boldsymbol{R}^3$. Furthermore, the dual total curvature satisfies not only the Cohn-Vossen inequality, but also the Osserman inequality [UY5, Yu2] (see also (3.13) in Section 3).

3. Basic preliminaries

Let $f\colon M \to H^3$ be a conformal CMC-1 immersion of a Riemann surface M into H^3. Let ds^2, dA and K denote the induced metric, induced area element and Gaussian curvature, respectively. Then $K \le 0$ and $d\sigma^2 := (-K)\,ds^2$ is a conformal pseudometric of constant curvature 1 on M. We call this pseudometric's developing map $g\colon \widetilde{M}(:=$ the universal cover of $M) \to \boldsymbol{CP}^1 = \boldsymbol{C} \cup \{\infty\}$ the *secondary Gauss map* of f. Namely, g is a conformal map so that its pull-back of the Fubini-Study metric of $\boldsymbol{CP}^1$ equals $d\sigma^2$:

$$d\sigma^2 = (-K)\,ds^2 = \frac{4\,dg\,d\bar{g}}{(1+g\bar{g})^2}\ . \tag{3.1}$$

Such a map g is determined by $d\sigma^2$ uniquely up to the change

$$g \mapsto a \star g := \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}}\ , \qquad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SU}(2)\ . \tag{3.2}$$

Since $d\sigma^2$ is invariant under the deck transformation group $\pi_1(M)$, there is a representation

$$\rho_g\colon \pi_1(M) \longrightarrow \mathrm{PSU}(2) \quad \text{such that} \quad g \circ \tau^{-1} = \rho_g(\tau) \star g \quad \bigl(\tau \in \pi_1(M)\bigr)\ , \tag{3.3}$$

where $\mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm \mathrm{id}\}$. The metric $d\sigma^2$ is called *reducible* if the image of ρ_g can be diagonalized simultaneously, and is called *irreducible* otherwise. In the case $d\sigma^2$ is reducible, we call it is $\mathcal{H}^3$*-reducible* if the image of ρ_g is the identity, and is called $\mathcal{H}^1$*-reducible* otherwise. We call a CMC-1 immersion $f\colon M \to H^3$ $\mathcal{H}^1$-reducible (resp. $\mathcal{H}^3$-reducible) if the corresponding pseudometric $d\sigma^2$ is $\mathcal{H}^1$-reducible (resp. $\mathcal{H}^3$-reducible). For details on reducibility, see [RUY1], for example.

In addition to g, two other holomorphic invariants G and Q are closely related to geometric properties of CMC-1 surfaces. The *hyperbolic Gauss map* $G\colon M \to \boldsymbol{CP}^1$ is holomorphic and is defined geometrically by identifying the ideal boundary of H^3 with $\boldsymbol{CP}^1$: $G(p)$ is the asymptotic class of the normal geodesic of $f(M)$ starting at $f(p)$ and oriented in the mean curvature vector's direction. The *Hopf differential* Q is a holomorphic symmetric 2-differential on M such that $-Q$ is the $(2,0)$-part of the complexified second fundamental form. The Gauss equation implies

$$ds^2 \cdot d\sigma^2 = 4\,Q \cdot \overline{Q}\ , \tag{3.4}$$

where $\cdot$ means the symmetric product. Moreover, these invariants are related by

$$S(g) - S(G) = 2Q\ , \tag{3.5}$$

where $S(\cdot)$ denotes the Schwarzian derivative:

$$S(h) := \left[\left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2\right] dz^2 \qquad \left({}' = \frac{d}{dz}\right)$$

with respect to a local complex coordinate z on M.

In terms of g and Q, the induced metric ds^2 and complexification of the second fundamental form h are

$$ds^2 = (1+|g|^2)^2 \left|\frac{Q}{dg}\right|^2 , \qquad h = -Q - \overline{Q} + ds^2 .$$

Since $K \le 0$, we can define the *total absolute curvature* as

$$\mathrm{TA}(f) := \int_M (-K)\, dA \in [0, +\infty] .$$

Then $\mathrm{TA}(f)$ is the area of the image of M in $\boldsymbol{CP}^1$ of the secondary Gauss map g. $\mathrm{TA}(f)$ is generally not an integer multiple of 4π — for catenoid cousins [Bry, Example 2] and their δ-fold covers, $\mathrm{TA}(f)$ admits *any* positive real number.

For each conformal CMC-1 immersion $f\colon M \to H^3$, there is a holomorphic null immersion $F\colon \widetilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$, the *lift* of f, satisfying the differential equation

$$dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega , \qquad \omega = \frac{Q}{dg} \tag{3.6}$$

so that $f = FF^*$, where $F^* = {}^t\overline{F}$ [Bry, UY1]. Here we consider

$$H^3 = \mathrm{SL}(2, \boldsymbol{C})/\,\mathrm{SU}(2) = \{aa^* \mid a \in \mathrm{SL}(2, \boldsymbol{C})\} .$$

We call a pair (g, ω) the *Weierstrass data* of f. The lift F is said to be *null* because $\det F^{-1}dF$, the pull-back of the Killing form of $\mathrm{SL}(2, \boldsymbol{C})$ by F, vanishes identically on M. Conversely, for a holomorphic null immersion $F\colon \widetilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$, $f := FF^*$ is a conformal CMC-1 immersion of $\widetilde{M}$ into H^3. If $F = (F_{ij})$, equation (3.6) implies

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}} , \tag{3.7}$$

and it is shown in [Bry] that

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}} . \tag{3.8}$$

The inverse matrix F^{-1} is also a holomorphic null immersion, and produces a new CMC-1 immersion $f^\# = F^{-1}(F^{-1})^*\colon \widetilde{M} \to H^3$, called the *dual* of f [UY5]. The induced metric $ds^{2\#}$ and the Hopf differential $Q^\#$ of $f^\#$ are

$$ds^{2\#} = (1+|G|^2)^2 \left|\frac{Q}{dG}\right|^2 , \qquad Q^\# = -Q . \tag{3.9}$$

So $ds^{2\#}$ and $Q^\#$ are well-defined on M itself, even though $f^\#$ might be defined only on $\widetilde{M}$. This duality between f and $f^\#$ interchanges the roles of the hyperbolic Gauss map G and secondary Gauss map g. In particular, one has

$$dF\, F^{-1} = -(F^{-1})^{-1} d(F^{-1}) = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG} . \tag{3.10}$$

Hence dFF^{-1} is single-valued on M, whereas $F^{-1}dF$ generally is not.

Since $ds^{2\#}$ is single-valued on M, we can define the *dual total absolute curvature*

$$\mathrm{TA}(f^\#) := \int_M (-K^\#)\, dA^\#,$$

where $K^{\#}$ (≤ 0) and $dA^{\#}$ are the Gaussian curvature and area element of $ds^{2\#}$, respectively. As

$$d\sigma^{2\#} := (-K^{\#})ds^{2\#} = \frac{4\,dG\,d\overline{G}}{(1+|G|^2)^2} \tag{3.11}$$

is a pseudo-metric of constant curvature 1 with developing map G, $\mathrm{TA}(f^{\#})$ is the area of the image of G on $\boldsymbol{CP}^1 = S^2$. The following assertion is important for us:

Lemma 3.1 ([UY5, Yu2])**.** *The Riemannian metric $ds^{2\#}$ is complete* (*resp. nondegenerate*) *if and only if ds^2 is complete* (*resp. nondegenerate*)*.*

We now assume that the induced metric ds^2 (and consequently $ds^{2\#}$) on M is complete and that either $\mathrm{TA}(f) < \infty$ or $\mathrm{TA}(f^{\#}) < \infty$, hence there exists a compact Riemann surface $\overline{M}_\gamma$ of genus γ and a finite set of points $\{p_1, \dots, p_n\} \subset \overline{M}_\gamma$ ($n \geq 1$) so that M is biholomorphic to $\overline{M}_\gamma \setminus \{p_1, \dots, p_n\}$ (see Theorem 9.1 of [Oss]). We call the points p_j the *ends* of f.

Unlike the Gauss map for minimal surface with $\mathrm{TA} < \infty$ in $\boldsymbol{R}^3$, the hyperbolic Gauss map G of the surface might not extend to a meromorphic function on $\overline{M}_\gamma$, as the Enneper cousin (Example 4.2) shows. However, the Hopf differential Q does extend to a meromorphic differential on $\overline{M}_\gamma$ [Bry]. We say an end p_j ($j = 1, \dots, n$) of a CMC-1 immersion is *regular* if G is meromorphic at p_j. When $\mathrm{TA}(f) < \infty$, an end p_j is regular precisely when the order of Q at p_j is at least -2, and otherwise G has an essential singularity at p_j [UY1]. Moreover, the pseudometric $d\sigma^2$ as in (3.1) has a *conical singularity* at each end p_j [Bry]. For a definition of conical singularity, see [UY3, UY7].

Analogue of the Osserman inequality. For a CMC-1 surface of genus γ with n ends, the second and third authors showed that the equality of the Cohn-Vossen inequality for the total absolute curvature never holds [UY1]:

$$\frac{1}{2\pi}\,\mathrm{TA}(f) > -\chi(M) = 2(\gamma - 2) + n\ . \tag{3.12}$$

The catenoid cousins (Example 4.3) show that this inequality is the best possible.

On the other hand, the dual total absolute curvature satisfies an Osserman-type inequality [UY5]:

$$\frac{1}{2\pi}\,\mathrm{TA}(f^{\#}) \geq -\chi(M) + n = 2(\gamma + n - 1)\ . \tag{3.13}$$

Moreover, equality holds exactly when all the ends are embedded: This follows by noting that equality is equivalent to all ends being regular and embedded ([UY5]), and that any embedded end must be regular (proved recently by Collin, Hauswirth and Rosenberg [CHR1] and independently by Yu [Yu3]).

Effects of transforming the lift $\boldsymbol{F}$. Here we consider the change $\hat{F} = aFb^{-1}$ of the lift F, where $a, b \in \mathrm{SL}(2, \boldsymbol{C})$. Then $\hat{F}$ is also a holomorphic null immersion, and the hyperbolic Gauss map $\hat{G}$, the secondary Gauss map $\hat{g}$ and the Hopf differential $\hat{Q}$ of $f = \hat{F}\hat{F}^*$ are given by (see [UY3])

$$\hat{G} = a \star G, \quad \hat{g} = b \star g, \quad \hat{Q} = Q\ . \tag{3.14}$$

In particular, the change $\hat{F} = aF$ moves the surface by a rigid motion of H^3, and does not change g and Q. Furthermore, for any $b \in \mathrm{SU}(2)$, the change $\hat{F} = Fb$ does not change the surface at all.

By choosing a suitable rigid motion $a \in \mathrm{SL}(2,\boldsymbol{C})$ of the surface in H^3, the expression for G can often be simplified, using

$$\hat{G} = a \star G = \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}} \;, \qquad (a_{ij})_{i,j=1,2} \in \mathrm{SL}(2,\boldsymbol{C}) \;. \tag{3.15}$$

4. Basic examples

Example 4.1 (Horosphere)**.** The horosphere is the only flat (and consequently totally umbilic) CMC-1 surface in H^3. It's Weierstrass data can be given by

$$g = 0, \qquad \omega = a\,dz \qquad (a \in \boldsymbol{C} \setminus \{0\}) \;.$$

The holomorphic lift $F\colon \boldsymbol{C} \to \mathrm{SL}(2,\boldsymbol{C})$ of the surface with initial condition $F(0) = \mathrm{id}$ is given by

$$F = \begin{pmatrix} 1 & 0 \\ az & 1 \end{pmatrix} \;.$$

In particular the hyperbolic Gauss map is a constant function, as well as the secondary Gauss map $g = 0$. So the total curvature and the dual total curvature of the horosphere are both equal to zero. Any flat or totally umbilic CMC-1 surfaces are parts of this surface. Planes in $\boldsymbol{R}^3$ are the corresponding minimal surfaces with the same Weierstrass data $(g,\omega) = (0, a\,dz)$.

Since the horosphere is simply-connected, it is $\mathcal{H}^3$-reducible.

Example 4.2 (Enneper cousin and dual of Enneper cousin)**.** The Enneper cousins are given in [Bry], with the same Weierstrass data as the Enneper surface in $\boldsymbol{R}^3$:

$$g = z, \qquad \omega = a\,dz \qquad \big(a \in \boldsymbol{C} \setminus \{0\}\big) \;.$$

Hence the Enneper cousins are isometric to minimal Enneper surfaces.

The holomorphic lift $F\colon \boldsymbol{C} \to \mathrm{SL}(2,\boldsymbol{C})$ of the surface with initial condition $F(0) = \mathrm{id}$ is given by

$$F = \begin{pmatrix} \cosh(az) & a^{-1}\sinh(az) - z\cosh(az) \\ a\sinh(az) & \cosh(az) - az\sinh(az) \end{pmatrix} \;.$$

In particular the hyperbolic Gauss map G is given by

$$G = a^{-1}\tanh(az) \;,$$

and hence the end at $z = \infty$ is irregular. So the Enneper cousins have complete induced metrics of total absolute curvature 4π and infinite dual total absolute curvature. If one takes the inverse of F, one gets the duals of the Enneper cousins. Since

$$Fd(F^{-1}) = -dFF^{-1} = \begin{pmatrix} -a\cosh(az)\sinh(az) & \sinh^2(az) \\ -a^2\cosh^2(az) & a\cosh(az)\sinh(az) \end{pmatrix} \;,$$

the Weierstrass data $(g^{\#}, \omega^{\#})$ of the dual of the Enneper cousin is given by

$$g^{\#} = a^{-1}\tanh(az), \qquad \omega^{\#} = a^2\cosh^2(az)\,dz \;.$$

This dual surface also has a complete induced metric, but now with infinite total absolute curvature (see Lemma 3.1).

Since the Enneper cousins and their duals are simply-connected, they are $\mathcal{H}^3$-reducible.

Example 4.3 (Catenoid cousins and warped catenoid cousins)**.** Here we describe the catenoid cousins and the warped catenoid cousins. The catenoid cousins are the only CMC-1 surfaces of revolution [Bry]. The warped catenoid cousins [UY1, RUY3] are less well known.

CMC-1 surfaces of genus 0 with two regular ends are classified in Theorem 6.2 in [UY1]. Here we describe a slightly refined version of this classification, which can also be found in [RUY4]: A complete conformal CMC-1 immersion $f\colon M = \boldsymbol{C}\setminus\{0\} \to H^3$ with regular ends has the following Weierstrass data

$$g = \frac{\delta^2 - l^2}{4l} z^l + b\ , \qquad \omega = z^{-l-1} dz\ , \tag{4.1}$$

with $l > 0$, $\delta \in \boldsymbol{Z}^+$, and $l \neq \delta$, and $b \geq 0$, where the case $b > 0$ occurs only when $l \in \boldsymbol{Z}^+$. When $b = 0$ and $\delta = 1$, the surface is called a *catenoid cousin*, which is rotationally symmetric. (The Weierstrass data of the catenoid cousin is often written as $g = z^\mu$ and $\omega = (1-\mu^2) z^{-\mu-1}\, dz/(4\mu)$. This is equivalent to (4.1) for $b = 0$ and $\delta = 1$ and $l = \mu$ by a coordinate change $z \mapsto ((1-\mu^2)/4\mu)^{(1/\mu)} z$.) Catenoid cousins are embedded when $0 < l < 1$ and have one circle of self-intersection when $l > 1$. When $b = 0$, f is a δ-fold cover of a catenoid cousin. When $b > 0$ (then automatically l is a positive integer), we call f a *warped catenoid cousin*, and its discrete symmetry group is the natural $\boldsymbol{Z}_2$ extension of the dihedral group D_l. Furthermore, the catenoid cousins and warped catenoid cousins can be written explicitly as

$$f = FF^*, \qquad F = F_0 B\ ,$$

where

$$F_0 = \sqrt{\frac{\delta^2 - l^2}{\delta}} \begin{pmatrix} \dfrac{1}{l-\delta} z^{(\delta-l)/2} & \dfrac{\delta - l}{4l} z^{(l+\delta)/2} \\ \dfrac{1}{l+\delta} z^{-(l+\delta)/2} & \dfrac{-(l+\delta)}{4l} z^{(l-\delta)/2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$

In particular, the hyperbolic Gauss map and Hopf differential are given by

$$G = z^\delta, \qquad Q = \frac{\delta^2 - l^2}{4z^2}\, dz^2\ ,$$

which are equal to the Gauss map and Hopf differential of (δ-fold covers of) the catenoids in $\boldsymbol{R}^3$. The dual total curvature of a catenoid cousin is 4π, but its total curvature is $4\pi l$, which can take any value in $(0, 4\pi) \cup (4\pi, \infty)$. On the other hand, the total absolute curvature and the dual total absolute curvature of warped catenoid cousins are always integer multiples of 4π.

The catenoid cousins are generally $\mathcal{H}^1$-reducible, except when l is an integer, in which case they are $\mathcal{H}^3$-reducible. The warped catenoid cousins are all $\mathcal{H}^3$-reducible.

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