BUBBLETONS IN 3-DIMENSIONAL SPACE FORMS VIA THE DPW METHOD

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Contents

1. Introduction

We define three-dimensional space forms as the unique complete simply-connected 3-dimensional Riemann manifolds $\boldsymbol{R}^3, H^3$ and S^3, of constant sectional curvature 0, -1 and 1, respectively. A more concrete description of space forms is given in Section 2.

First we briefly explain the meaning of constant mean cuvature (CMC) here: We consider a smooth orientable surface $\mathcal{F}$ in a three-dimensional space form, and we fix a direction for the unit normal field N on $\mathcal{F}$. N is called the Gauss map of the surface. At each point $P \in \mathcal{F}$ we can consider the minimum and maximum of the directional curvatures (the eigenvalues of the operator dN on the tangent space), and these are the principal curvatures at P. We set H to be the average of the principal curvatures and call it the mean curvature. We call $\mathcal{F}$ a constant mean curvature (CMC) surface if H is the same at every point on $\mathcal{F}$.

It is well known that soap films have the property of attaining the least area with respect to the fixed volumes they bound, and soap films are examples of CMC surfaces. Mathematically, we have the following property:

> H = constant $\Longrightarrow$ compact portions of the surface are critical values for boundary-preserving, volume-preserving variations.

The sphere is a simple example of a CMC surface because all principal curvatures are the same on the sphere. The sphere is an important example because it is a closed CMC surface. For a long time, there were no known closed CMC surfaces besides the sphere, and we had the following Hopf conjecture:

> There are no closed CMC surfaces different from the standard sphere.

This conjecture had been thought to be correct, because:

(1) Hopf showed that the only genus-zero closed CMC surfaces in $\boldsymbol{R}^3$ are spheres.
(2) Alexandrov showed that the only embedded closed CMC surfaces in $\boldsymbol{R}^3$ are spheres.

However, Wente showed existence of immersed CMC tori in 1984 and this led to renewed interest in the field. Then Wente, Abresch, Pinkall, Sterling, and Bobenko found all the CMC tori in $\boldsymbol{R}^3$.

With this renewed interest, bubbletons in $\boldsymbol{R}^3$ have been closely examined in [26], [17] and [24]. In this paper, we analyze bubbleton surfaces in all three space forms $\boldsymbol{R}^3$ and S^3 and H^3, using the DPW method. Bubbleton surfaces are CMC surfaces made from Bäcklund transformations (in Bianchi's sense) of round cylinders. The surface is shaped like a cylinder with attached bubbles, thus it is called a bubbleton. The parallel constant positive Gaussian curvature surface of the bubbleton is well known. It was first found by Sievert [23], thus it is called the Sievert surface.

With respect to the DPW method, the Bäcklund transformation is a dressing action on loop groups and this dressing action is described by elements of the simplest possible type like those of Terng and Uhlenbeck [25]. Using these elements

> we find an explicit immersion formula and solve the period problems for bubbletons in $\boldsymbol{R}^3$ and S^3 and H^3.

More generally, we can do the Bäcklund transformation for any surfaces. So

> we can solve period problems for the Bäcklund transformations of general Delaunay surfaces, which we do here.

In the $\boldsymbol{R}^3$ case, this is also done in [26], [17].

In order to apply the DPW method, we first note that classical surface theory can be rewritten in modern fashion using quaternions. More concretely, we write quaternions using 2×2 matrices and identify 3-dimensional Euclidean space with the space of imaginary quaternions. Then the classical surface theory can be described using 2×2 matrices. For CMC surfaces, the Gauss-Codazzi equations (G-C) are the compatibility conditions for a system of equations of Lax pair type and allow a one-parameter family of deformations preserving H and the metric that changes only the Hopf differential. (This parameter is called the spectral parameter. Existence of this spectral parameter means we can say that we are working with an integrable equation (see [3]).) In this framework we have a CMC immersion formula called the Sym-Bobenko formula. The solutions of this Lax pair are 2×2 special unitary matrices in $SU(2)$, and we call them the extended frames for the CMC surfaces, and they are used in the Sym-Bobenko formula. More concretely, this $SU(2)$ solution depends on the spectral parameter and thus lies in the loop group $\Lambda SU(2)$. The spectral parameter then becomes the loop parameter.

The DPW method was created by Dorfmeister and Pedit and Wu (see [8]) for making CMC surfaces in $\boldsymbol{R}^3$. The DPW method uses loop group theory involving the loop groups $\Lambda SL(2, \boldsymbol{C})$, $\Lambda SU(2)$ and $\Lambda_+ SL(2, \boldsymbol{C})$ to be defined later and is related to the methods of integrable systems. The DPW method also (equivalently) makes extended frames corresponding to harmonic maps from Riemann surfaces to the unit sphere S^2. Using holomorphic 1-forms, the DPW method constructs holomorphic maps to $\Lambda SL(2, \boldsymbol{C})$ and after that constructs extended frames corresponding to harmonic maps. More concretely, one first gives $\Lambda sl(2, \boldsymbol{C})$-matrix-valued holomorphic 1-forms called holomorphic potentials. Next one solves a linear first-order (homogeneous) ordinary differential equation whose coefficient is the above holomorphic potential. The solution of this equation is in $\Lambda SL(2, \boldsymbol{C})$ when the initial condition is chosen in $\Lambda SL(2, \boldsymbol{C})$. We then decompose $\Lambda SL(2, \boldsymbol{C})$ to $\Lambda SU(2) \times \Lambda SL_+(2, \boldsymbol{C})$ via Iwasawa splitting, producing an $\Lambda SU(2)$ element from an $\Lambda SL(2, C)$ element. This element in $\Lambda SU(2)$ is an extended frame of a CMC surface. Finally, the Sym-Bobenko formula produces the CMC immersion. The advantages of this DPW approach are that we can deal with the asymptotic behaviors and period problems for CMC surfaces.

The paper is organized as follows: In Section 2, we give basic notations and results for all space forms CMC surfaces using 2 by 2 matrices. In Section 3, we give the DPW method. In Section 4, we give simple examples (cylinders and Delaunay surfaces) by the DPW construction. In Section 5, we give the construction and explicit parametrization of CMC bubbletons using simple examples from Section 4, and we show equivalence of the simple type dressing and Bianchi's Bäcklund transformation, and we consider the parallel surface of the cylinder bubbletons.

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2. Lax pairs for CMC surfaces in space forms

The arguments in this section are similar to arguments in [1] and [19].

2.1. The space forms. S^3, resp. H^3, is the unique complete simply connected 3-dimensional Riemannian manifold with constant sectional curvature $+1$, resp. -1.

There are a variety of models for describing S^3 and H^3. S^3 is the unit 3-sphere in $\boldsymbol{R}^4$ with the metric induced by $\boldsymbol{R}^4$, but for viewing graphics of CMC surfaces in S^3, we shall stereographically project S^3 from its north pole to the space $\boldsymbol{R}^3 \cup \{\infty\}$. For H^3 we shall use the Lorentz model:

$$H^3 = \{(t, x, y, z) \in R^{3,1} \,|\, x^2 + y^2 + z^2 - t^2 = -1\,,\, t > 0\}$$

with the metric induced by $\boldsymbol{R}^{3,1}$, where $\boldsymbol{R}^{3,1}$ is the 4-dimensional Lorentz space

$$\{(t, x, y, z) \,|\, t, x, y, z \in \boldsymbol{R}\}$$

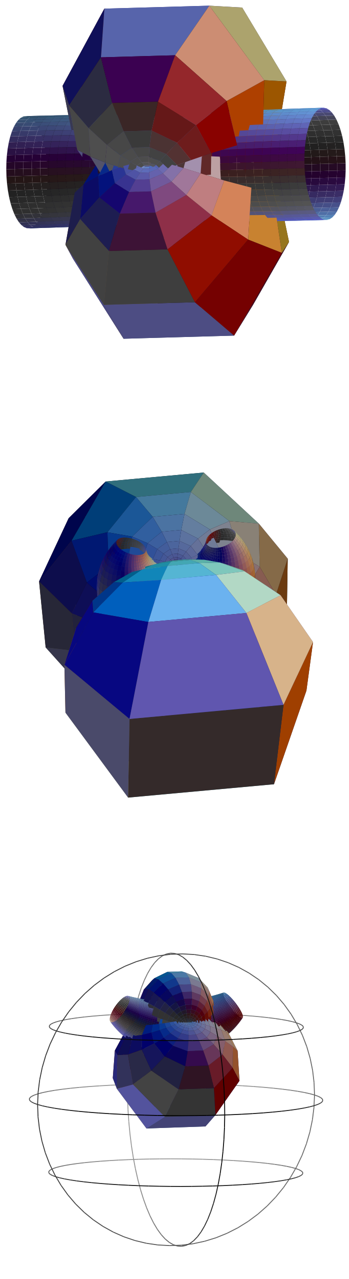

FIGURE 1. CMC bubbletons in R^3, S^3 and H^3. The R^3 bubbleton was first described in [26]. (The S^3 graphic here was made by N. Schmitt [22])

with the Lorentz metric

$$\langle (t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2 - t_1 t_2 .$$

This metric is not positive definite, but its restriction to the tangent space of H^3 is positive definite. For viewing graphics of CMC surfaces in H^3, we shall use the Poincare model for H^3, which is stereographic projection of the Minkowski model in Lorentz space from the point $(0,0,0,-1)$ to the 3-ball $\{(0,x,y,z) \in R^{3,1} \,|\, x^2 + y^2 + z^2 < 1\} \cong \{p = (x,y,z) \in \boldsymbol{R}^3 \,|\, |p| < 1\}$.

2.2. **Surfaces in the space forms.** Before we describe the DPW representation for CMC surfaces in Section 3, we show here that CMC surfaces in 3-dimensional space forms are locally equivalent to solutions of a certain kind of Lax pair. Then proving that the DPW recipe gives all CMC surfaces means showing that it gives all possible solutions for this certain kind of Lax pair.

Let M be a Riemann surface and let $f : M \to \mathcal{M}^3$ be a CMC conformal immersion, where $\mathcal{M}^3$ is either $\boldsymbol{R}^3$ or S^3 or H^3. Let Σ be a simply-connected domain in M with conformal coordinate $z = x + iy$ defined on Σ. We can consider the restriction $f|_\Sigma$ of f to Σ, i.e.

$$f = f(z, \bar{z}) : \Sigma \to \mathcal{M}^3 = \boldsymbol{R}^3 \text{ or } S^3 \text{ or } H^3 .$$

We write f as a function of both z and $\bar{z}$ to emphasize that f is not holomorphic in z.

Each of the three space forms lies isometrically in a vector space V: V is just $\boldsymbol{R}^3$ in the case $\mathcal{M}^3 = \boldsymbol{R}^3$; $V = \boldsymbol{R}^4$ in the case $\mathcal{M}^3 = S^3$; and $V = \boldsymbol{R}^{3,1}$ in the case $\mathcal{M}^3 = H^3$. Let $\langle \cdot, \cdot \rangle$ be the inner product associated to V, which is the Euclidean inner product in the first two cases, and the Lorentz inner product in the third case. We may also view f as a map into V, i.e.

$$f : \Sigma \to \mathcal{M}^3 \subseteq V = \boldsymbol{R}^3 \text{ or } \boldsymbol{R}^4 \text{ or } \boldsymbol{R}^{3,1} .$$

The derivatives $f_x = \partial_x f$ and $f_y = \partial_y f$ are vectors in the tangent space $T_{f(z,\bar{z})}V$ of V at $f(z,\bar{z})$. Because V is a vector space, V naturally corresponds to $T_{f(z,\bar{z})}V$, so f_x and f_y can be viewed as lieing in V itself. So $f_z = (1/2)(f_x - i f_y)$ and $f_{\bar{z}} = (1/2)(f_x + i f_y)$ are defined in the complex extension $V_{\boldsymbol{C}} = \{c \cdot v \,|\, c \in \boldsymbol{C}, v \in V\}$ of V with inner product extended to $\langle c_1 v_1, c_2 v_2 \rangle = c_1 c_2 \langle v_1, v_2 \rangle$ (which we also denote by $\langle \cdot, \cdot \rangle$ and is not a true inner product on $V_{\boldsymbol{C}}$). Since f is conformal, we have

$$\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0 \; , \;\; \langle f_z, f_{\bar{z}} \rangle = 2e^{2u} \; ,$$

where the right-most equation defines the function $u : \Sigma \to \boldsymbol{R}$.

There is a natural notion of a unit normal vector $N = N(z,\bar{z}) \in T_{f(z,\bar{z})}V \equiv V$ of f, defined by the properties

(1) $\langle N, N \rangle = 1$,
(2) $N \in T_{f(z,\bar{z})}\mathcal{M}^3$, and
(3) $\langle N, f_z \rangle = \langle N, f_{\bar{z}} \rangle = 0$.

In each space form, the mean curvature of f is given by

$$H = \frac{1}{2e^{2u}} \langle f_{z\bar{z}}, N \rangle \; , \tag{2.1}$$

which is constant, by assumption. We also define the Hopf differential to be

$$Q = \langle f_{zz}, N \rangle \; .$$

Because f exists as a surface in $\mathcal{M}^3$, u and H and Q satisfy the Gauss and Codazzi equations for $\mathcal{M}^3$. For H constant, we will see that the Gauss and Codazzi equations for $\mathcal{M}^3$ remain satisfied when Q is replaced by $\lambda^{-2}Q$ for any $\lambda \in S^1 = \{p \in \boldsymbol{C} \mid |p| = 1\}$. Hence, up to rigid motions, there is a unique surface f_λ with metric determined by u and with mean curvature H and Hopf differential $\lambda^{-2}Q$. (We use the notation f_λ to state that f depends on λ; it does *not* denote the derivative $\partial_\lambda f$.) The surfaces f_λ for $\lambda \in S^1$ form a one-parameter family called the *associate family* of f. The parameter λ is called the *spectral parameter* and is essential to the DPW method.

We remark that in the cases of S^3 and H^3, we will actually be choosing Q so that it differs from the true Hopf differential by a particular constant factor.

2.3. **The vector spaces V in terms of quaternions.** Define the matrices

$$\sigma_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We can think of $\mathcal{Q} = \mathrm{span}_{\boldsymbol{R}}\{i\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3\}$ as the quaternions because it has the quaternionic algebraic structure.

2.3.1. When $\mathcal{M}^3 = V = \boldsymbol{R}^3$, we associate $\mathcal{M}^3$ with the imaginary quaternions $\mathcal{Q}_{Im} = \mathrm{span}_{\boldsymbol{R}}\{-i\sigma_1, -i\sigma_2, -i\sigma_3\} \subseteq \mathcal{Q}$ by the map

$$(x_1, x_2, x_3) \to x_1 \frac{i}{2}\sigma_1 + x_2 \frac{i}{2}\sigma_2 + x_3 \frac{i}{2}\sigma_3 .$$

Then for $X, Y \in \mathcal{Q}_{Im}$, the inner product inherited from $\boldsymbol{R}^3$ is

$$\langle X, Y \rangle = -2 \cdot \mathrm{trace}(XY) = +2 \cdot \mathrm{trace}(XY^*) , \tag{2.2}$$

where $Y^* := \bar{Y}^t$. Also, any oriented orthonormal basis $\{X, Y, Z\}$ of vectors of $\mathcal{M}^3 \equiv \mathcal{Q}_{Im}$ satisfies

$$X = F\left(\frac{i}{2}\sigma_1\right)F^{-1} , \quad Y = F\left(\frac{i}{2}\sigma_2\right)F^{-1} , \quad Z = F\left(\frac{i}{2}\sigma_3\right)F^{-1} \tag{2.3}$$

for some $F \in \mathrm{SU}(2)$, and this F is unique up to sign. In other words, rotations of $\boldsymbol{R}^3$ fixing the origin are represented in the quaternionic representation $\mathcal{Q}_{Im}$ of $\boldsymbol{R}^3$ by matrices $F \in \mathrm{SU}(2)$.

2.3.2. When $\mathcal{M}^3 = S^3$ and $V = \boldsymbol{R}^4$, we associate V with $\mathcal{Q}$ by the map

$$(x_1, x_2, x_3, x_4) \to x_1 i\sigma_0 + x_2 i\sigma_1 + x_3 i\sigma_2 + x_4 i\sigma_3 ,$$

so points $(x_1, x_2, x_3, x_4) \in V = \boldsymbol{R}^4$ are matrices of the form

$$X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} , \tag{2.4}$$

where $a = x_1 + ix_4$ and $b = x_3 + ix_2$. That is, they are matrices X that satisfy

$$X = \sigma_2 \bar{X} \sigma_2 . \tag{2.5}$$

The inner product on $\mathcal{Q}$ inherited from V is

$$\langle X, Y \rangle = (1/2) \cdot \mathrm{trace}(XY^*) , \tag{2.6}$$

where $Y^* := \bar{Y}^t$. Note that this inner product is the same as in (2.2), up to a factor of 4.

2.3.3. When $\mathcal{M}^3 = H^3$ and $V = \boldsymbol{R}^{3,1}$, we can associate V with the set of self-adjoint matrices $\{X \in M_{2\times 2} \,|\, X^* = X\}$ by the map

$$(x_0, x_1, x_2, x_3) \in R^{3,1} \to X = x_0 i\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \ .$$

One can check that $\sigma_2 X^t \sigma_2 = X^{-1}\det X$ and the inner product inherited from V is

$$\langle X, Y\rangle = (-1/2)\mathrm{trace}(X\sigma_2 Y^t \sigma_2) \ ,$$

so

$$\langle X, X\rangle = -\det X \ ,$$

for self-adjoint matrices X, Y.

2.4. **The case $\mathcal{M}^3 = \boldsymbol{R}^3$.** Consider $\mathcal{M}^3 = V = \boldsymbol{R}^3$ in the quaternionic representation, with inner product $\langle\cdot,\cdot\rangle$ as in Section 2.3. Let $f : M \to \boldsymbol{R}^3$ be a CMC H conformal immersion, and let f_λ be as in Section 2.2. By applying a homothety if necessary, we may assume $H = 1/2$. So

$$\langle (f_\lambda)_{z\bar{z}}, N\rangle = e^{2u} \ . \tag{2.7}$$

Note that N here also now depends on λ. We have

$$\langle (f_\lambda)_z, (f_\lambda)_{zz}\rangle = \langle (f_\lambda)_z, (f_\lambda)_{z\bar{z}}\rangle = \langle N_z, N\rangle = 0 \ , \quad \langle (f_\lambda)_{\bar{z}}, (f_\lambda)_{zz}\rangle = 4u_z e^{2u} \ ,$$
$$\langle N_z, (f_\lambda)_{\bar{z}}\rangle = -\langle N, (f_\lambda)_{z\bar{z}}\rangle = -e^{2u} \ , \quad \langle N_z, (f_\lambda)_z\rangle = -\lambda^{-2}Q \ .$$

So

$$(f_\lambda)_{zz} = 2u_z(f_\lambda)_z + \lambda^{-2}QN \ , \quad (f_\lambda)_{z\bar{z}} = e^{2u}N \ , \quad (f_\lambda)_{\bar{z}\bar{z}} = 2u_{\bar{z}}(f_\lambda)_{\bar{z}} + \lambda^2\bar{Q}N,$$
$$N_z = \frac{1}{2}(-(f_\lambda)_z - \lambda^{-2}Qe^{-2u}(f_\lambda)_{\bar{z}}) \ , \quad N_{\bar{z}} = \frac{1}{2}(-(f_\lambda)_{\bar{z}} - \lambda^2\bar{Q}e^{-2u}(f_\lambda)_z) \ .$$

With θ defined by $\lambda = e^{i\theta}$, we define

$$e_1 = -\frac{(f_\lambda)_x}{|(f_\lambda)_x|}\cos\theta - \frac{(f_\lambda)_y}{|(f_\lambda)_y|}\sin\theta \ , \quad e_2 = -\frac{(f_\lambda)_x}{|(f_\lambda)_x|}\sin\theta + \frac{(f_\lambda)_y}{|(f_\lambda)_y|}\cos\theta \ .$$

Therefore

$$(f_\lambda)_x = -|(f_\lambda)_x|(e_1\cos\theta + e_2\sin\theta) \ , \quad (f_\lambda)_y = -|(f_\lambda)_y|(e_1\sin\theta - e_2\cos\theta)$$

Since e_1, e_2 and N form an oriented orthonormal frame in $\boldsymbol{R}^3$, there exists an $F \in \mathrm{SU}(2)$ as in (2.3), depending on z, $\bar{z}$, and also on λ, such that

$$e_1 = F\left(\frac{i}{2}\sigma_1\right)F^{-1} \ , \quad e_2 = F\left(\frac{i}{2}\sigma_2\right)F^{-1} \ , \quad N = F\left(\frac{i}{2}\sigma_3\right)F^{-1} \ . \tag{2.8}$$

Note that here we are now writing e_1, e_2, and N in quaternionic form, rather than in the vector form as above.

By applying a rigid motion to f_λ for each $\lambda \in S^1$ if necessary, we can fix $F(z_*, \bar{z}_*, \lambda) = \mathrm{id}$ for all $\lambda \in S^1$ for some fixed $z_* \in \Sigma$, i.e. we may assume that the vectors e_1, e_2, and N are the unit vectors in the positive x-axis, y-axis, and z-axis directions, respectively, at the point z_* for all $\lambda \in S^1$.

We define

$$U := F^{-1}F_z \ , \quad V := F^{-1}F_{\bar{z}} \ .$$

By (2.8), we have

$$(f_\lambda)_z = -ie^u F\begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix}F^{-1} \ , \quad (f_\lambda)_{\bar{z}} = -ie^u F\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}F^{-1} \ . \tag{2.9}$$

Then $(f_\lambda)_{z\bar{z}} = (f_\lambda)_{\bar{z}z}$ implies that $V_{11} - V_{22} = -u_{\bar{z}}$ and $U_{22} - U_{11} = -u_z$ and $\lambda^{-1}V_{21} = -\lambda U_{12}$. Furthermore, $(f_\lambda)_{z\bar{z}} = e^{2u}N$ implies $V_{21} = \lambda e^u/2$ and $(f_\lambda)_{zz} = 2u_z(f_\lambda)_z + \lambda^{-2}QN$ implies $U_{21} = e^{-u}\lambda^{-1}Q/2$ and $(f_\lambda)_{\bar{z}\bar{z}} = 2u_{\bar{z}}(f_\lambda)_{\bar{z}} + \lambda^2\bar{Q}N$ implies $V_{12} = -e^{-u}\lambda\bar{Q}/2$. Since $\det F = 1$, U and V are traceless, so U and V are as in equation (2.12).

The compatibility condition for the existence of a solution F to the Lax pair (2.11) is $F_{z\bar{z}} = F_{\bar{z}z}$, in other words,

$$U_{\bar{z}} - V_z - [U, V] = 0$$

(also called the Maurer-Cartan equation) and this is equivalent to (2.10), which is the Gauss and Codazzi equations for f. (So H=constant implies Q is holomorphic.)

Note that (2.10) still holds if Q is replaced with $\lambda^{-2}Q$ for any $\lambda \in S^1$, which is what allowed us to consider the entire associate family f_λ.

We have now seen that any CMC 1/2 surface f gives a solution to the Lax pair (2.11) with U and V as in (2.12). One can check that the surface defined by the Sym-Bobenko formula (2.13) has the same derivatives with respect to z and $\bar{z}$ as f does in (2.9) at $\lambda = 1$. Hence the surface in (2.13) is the surface f, up to a translation.

Conversely, starting with u and Q satisfying (2.10) and taking a solution F of (2.11)-(2.12), note that f in (2.13) is a conformal CMC 1/2 immersion with metric $ds^2 = 4e^{2u}(dx^2 + dy^2)$ and Hopf differential Q (because f_z and $f_{\bar{z}}$ satisfy (2.9) at $\lambda = 1$. Furthermore, u satisfies (2.7), so by (2.1), $H = \frac{1}{2}$ (using (2.12)).

We conclude that finding arbitrary CMC 1/2 surfaces is equivalent to finding arbitrary integrable Lax pairs of the form (2.11) and their solutions, with U and V as in (2.12). Thus we have proven the following theorem, whose notations match those of [8] and [9] and [15].

Theorem 2.1. *Let u and Q solve*

$$4u_{z\bar{z}} - Q\bar{Q}e^{-2u} + e^{2u} = 0\ , \quad Q_{\bar{z}} = 0\ , \tag{2.10}$$

and let $F(z, \bar{z}, \lambda)$ be a solution, which is in $\mathrm{SU}(2)$ for all $\lambda \in S^1$ and is complex analytic in λ, of the system

$$F_z = FU\ , \quad F_{\bar{z}} = FV \tag{2.11}$$

with

$$U = \frac{1}{2}\begin{pmatrix} u_z & -e^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix}\ , \quad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u}\lambda \\ e^u\lambda & u_{\bar{z}} \end{pmatrix}\ . \tag{2.12}$$

Define

$$f = \left[\frac{-1}{2}F\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}F^{-1} - i\lambda(\partial_\lambda F)\cdot F^{-1}\right]\Bigg|_{\lambda=1}\ . \tag{2.13}$$

Then f is of the form

$$\frac{-i}{2}\begin{pmatrix} -t & r+is \\ r-is & t \end{pmatrix}\ , \tag{2.14}$$

for reals r, s, t, and

$$(r, s, t)$$

is a conformal parametrization of a CMC $1/2$ surface in $\boldsymbol{R}^3$, parametrized by z. Furthermore, every CMC $1/2$ conformal immersion in $\boldsymbol{R}^3$ can be attained this way.

2.5. **The case $\mathcal{M}^3 = S^3$.** Consider a conformal CMC H immersion $f: M \to \mathcal{M}^3 = S^3 \subset V = \boldsymbol{R}^4$ of a Riemann surface M, with $\langle\cdot,\cdot\rangle$ as in (2.6). Then

$$\langle f, f\rangle = 1\ .$$

Let us change the name of the local complex coordinate z to w, and let us change the name of Q to A. We have

$$\langle f_w, f_{\bar w}\rangle = 2e^{2u}\ ,\quad \langle f, N\rangle = \langle f_w, N\rangle = \langle f_{\bar w}, N\rangle = 0\ ,$$
$$\langle N, N\rangle = 1\ ,\quad A = \langle f_{ww}, N\rangle\ .$$

One can show that if $\sigma = (f, f_w, f_{\bar w}, N)^t$, then σ satisfies

$$\sigma_w = \mathcal{U}\sigma\ ,\quad \sigma_{\bar w} = \mathcal{V}\sigma\ ,$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2u_w & 0 & A \\ -2e^{2u} & 0 & 0 & 2He^{2u} \\ 0 & -H & \frac{-1}{2}Ae^{-2u} & 0 \end{pmatrix}\ ,\quad \mathcal{V} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2e^{2u} & 0 & 0 & 2He^{2u} \\ 0 & 0 & 2u_{\bar w} & \bar A \\ 0 & \frac{-1}{2}\bar A e^{-2u} & -H & 0 \end{pmatrix}\ .$$

Since H is constant, the compatibility condition

$$\mathcal{U}_{\bar w} - \mathcal{V}_w + [\mathcal{U}, \mathcal{V}] = 0\ ,$$

implies that

$$2u_{w\bar w} + 2e^{2u}(1+H^2) - \frac{1}{2}A\bar A e^{-2u} = 0\ ,\quad A_{\bar w} = 0\ .$$

Making the change of variables $z = 2\sqrt{1+H^2}w$ and $A = 2\sqrt{1+H^2}e^{2i\psi}Q$ for a real constant ψ, we find that (2.10) holds. The conclusion is that a CMC H conformal immersion f into S^3 with conformal parameter w and metric $4e^{2u}dwd\bar w$ has another conformal parameter z such that $Q = 2\sqrt{1+H^2}e^{-2i\psi}\langle f_{zz}, N\rangle$ is holomorphic and u satisfies the first equation in (2.10). Note that, with respect to the parameter z, the metric is not represented by u (in the sense that $4e^{2u}dzd\bar z$ is *not* the metric) and Q is not the Hopf differential (Q differs from the true Hopf differential by a constant factor).

We now consider how to get a surface from given u, Q, and H.

Theorem 2.2. *Let u and Q solve* (2.10) *and let $F_j(z,\bar z,\lambda = e^{-i\gamma_j})$, $j = 1, 2$, be two solutions of the system* (2.11)-(2.12) *such that $F(z,\bar z,\lambda) \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and $F(z,\bar z,\lambda)$ is complex analytic in λ. Define*

$$f = F_1 \begin{pmatrix} \sqrt{e^{i(\gamma_2-\gamma_1)}} & 0 \\ 0 & \sqrt{e^{i(\gamma_1-\gamma_2)}} \end{pmatrix} F_2^{-1}\ . \tag{2.15}$$

Then f is a conformal immersion with CMC $H = \cot(\gamma_1 - \gamma_2)$ into S^3. Conversely, every conformal immersion with CMC $H = \cot(\gamma_1 - \gamma_2)$ into S^3 can be attained this way.

Proof. We will use

$$N = iF_1 \begin{pmatrix} \sqrt{e^{i(\gamma_2-\gamma_1)}} & 0 \\ 0 & -\sqrt{e^{i(\gamma_1-\gamma_2)}} \end{pmatrix} F_2^{-1}$$

here. We will also need to consider the inner product more generally as: $\langle X, Y\rangle = (1/2)\cdot \mathrm{trace}(X\sigma_2 Y^t \sigma_2)$, not $\langle X, Y\rangle = (1/2)\cdot\mathrm{trace}(XY^*)$.

Show $\sigma_2 \bar{f} \sigma_2 = f$ and $\sigma_2 \bar{N} \sigma_2 = N$, so $f, N \in \boldsymbol{R}^4$. Note that $F \in \mathrm{SU}(2)$ implies $(1/2)\mathrm{trace}(ff^*) = 1$, so $f \in S^3$. Also, verify

$$\langle f_z, N\rangle = \langle f_{\bar{z}}, N\rangle = \langle f, N\rangle = \langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0 ,$$

and

$$\langle N, N\rangle = 1 , \quad \langle f_z, f_{\bar{z}}\rangle = \frac{1}{2} e^{2u} \sin^2(\gamma_1 - \gamma_2) , \quad \langle f_{zz}, N\rangle = \frac{1}{2} Q e^{2i\psi} \sin(\gamma_1 - \gamma_2) ,$$

and

$$f_{z\bar{z}} = \frac{-1}{2} e^{2u} \sin^2(\gamma_1 - \gamma_2) f + \frac{1}{2} e^{2u} \sin(\gamma_1 - \gamma_2) \cos(\gamma_1 - \gamma_2) N .$$

If H is given by $H = \cot(\gamma_1 - \gamma_2)$, $z = 2\sqrt{1+H^2}w = 2w/\sin(\gamma_1 - \gamma_2)$, $A = \langle f_{ww}, N\rangle = (2\sqrt{1+H^2})^2 \langle f_{zz}, N\rangle = 2Qe^{2i\psi}/\sin(\gamma_1 - \gamma_2)$, then $\sigma_w = \mathcal{U}\sigma$, $\sigma_{\bar{w}} = \mathcal{V}\sigma$ hold.

To prove the converse, one can argue as we did in the $\boldsymbol{R}^3$ case. □

2.6. **The case $\mathcal{M}^3 = H^3$, with $H > 1$.** Let f be a conformally immersed CMC $H > 1$ surface in H^3 with the name of the local coordinate changed from z to w. We have

$$0 = \langle N, f\rangle = \langle f, N\rangle = \langle f_w, N\rangle = \langle f_{\bar{w}}, N\rangle , \quad \langle N, N\rangle = 1 ,$$
$$\langle f_w, f_{\bar{w}}\rangle = 2e^{2u} , \quad A = \langle f_{ww}, N\rangle .$$

Let us also change the name of Q to A. Let $\sigma = (f, f_w, f_{\bar{w}}, N)^t$, one can show that

$$\sigma_w = \mathcal{U}\sigma , \quad \sigma_{\bar{w}} = \mathcal{V}\sigma ,$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2u_w & 0 & A \\ 2e^{2u} & 0 & 0 & 2He^{2u} \\ 0 & -H & \frac{-1}{2}Ae^{-2u} & 0 \end{pmatrix} , \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2e^{2u} & 0 & 0 & 2He^{2u} \\ 0 & 0 & 2u_{\bar{w}} & \bar{A} \\ 0 & \frac{-1}{2}\bar{A}e^{-2u} & -H & 0 \end{pmatrix} .$$

Since $H\rangle 1$ is constant, the compatibility condition

$$\mathcal{U}_{\bar{w}} - \mathcal{V}_w + [\mathcal{U}, \mathcal{V}] = 0 ,$$

implies that

$$2u_{w\bar{w}} + 2e^{2u}(H^2 - 1) - \frac{1}{2}A\bar{A}e^{-2u} = 0 , \quad A_{\bar{w}} = 0 .$$

Making the change of variables $z = 2\sqrt{H^2 - 1}w$ and $A = 2\sqrt{H^2 - 1}e^{2i\psi}Q$ for a real constant ψ, we have that (2.10) holds. The conclusion is that a CMC H conformal immersion into H^3 with $H > 1$ has a conformal parameter z such that $Q = 2\sqrt{H^2 - 1}e^{-2i\psi}\langle f_{zz}, N\rangle$ is holomorphic and u satisfies the first equation in (2.10).

We now consider how to get a surface from given u, Q, and H.

Theorem 2.3. *Let u and Q solve* (2.10) *and let $F(z, \bar{z}, \lambda = e^{-q/2}e^{-i\psi})$ for some real q be a solution of the system* (2.11)-(2.12) *such that $F \in \mathrm{SU}(2)$ for all $\lambda \in S^1$ and F is complex analytic in λ. Then*

$$f = F \begin{pmatrix} 0 & -ie^{-q/2} \\ ie^{q/2} & 0 \end{pmatrix} \overline{F^{-1}} \sigma_2 \tag{2.16}$$

is a CMC $H = \coth q$ conformal immersion into H^3. Conversely, all CMC $H = \coth q$ conformal immersions into H^3 can be attained this way.

Proof. We will need

$$N = F \begin{pmatrix} 0 & -ie^{-q/2} \\ -ie^{q/2} & 0 \end{pmatrix} \bar{F}^{-1}\sigma_2$$

here.

Using the property $\sigma_2 X^t \sigma_2 = X^{-1} \det X$, check that

$$\langle N, f\rangle = \langle N, f_z\rangle = \langle N, f_{\bar z}\rangle = \langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0$$

and

$$\langle N, N\rangle = 1\ , \quad \langle f_z, f_{\bar z}\rangle = (1/2)e^{2u}\sinh^2 q\ .$$

Also

$$\langle f_{zz}, N\rangle = \frac{1}{2} Q e^{2i\psi} \sinh q\ ,$$

and

$$f_{z\bar z} = \frac{1}{2}e^{2u}\sinh^2 q f + \frac{1}{2}e^{2u}\sinh q \cosh q N\ .$$

With $z = 2\sqrt{H^2-1}w$, $A = 2\sqrt{H^2-1}e^{2i\psi}Q$, and $\sinh q = 1/\sqrt{H^2-1}$, then $A = \langle f_{ww}, N\rangle$ and $f_{w\bar w} = 2e^{2u}f + 2e^{2u}\coth q N$, so $H = \coth q$.

Note that

$$f = F(\lambda)^{-1}\begin{pmatrix} e^{-q/2} & 0 \\ 0 & e^{q/2}\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\overline{F(\lambda)}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} =$$

$$= F^{-1}\begin{pmatrix} e^{-q/2} & 0 \\ 0 & e^{q/2}\end{pmatrix}(F^*)^{-1}$$

$$= \frac{1}{|ad-bc|^2}\begin{pmatrix} e^{-q/2}d\bar d + e^{q/2}b\bar b & -\bar c d e^{-q/2} - \bar a b e^{q/2} \\ -c\bar d e^{-q/2} - a\bar b e^{q/2} & c\bar c e^{-q/2} + a\bar a e^{q/2}\end{pmatrix}\ ,$$

where

$$F = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\ .$$

Hence the diagonal of f is real and the off-diagonal terms are conjugate. Check that $\langle f, f\rangle = -1$, using $\sigma_2 X \sigma_2 = (X^{-1})^t$.

The converse is proven similarly to the argument in the $\boldsymbol{R}^3$ case. □

3. The DPW recipe

We saw in Section 2 that finding CMC $H \neq 0$ surfaces in $\boldsymbol{R}^3$ and CMC H surfaces in S^3 and CMC $H > 1$ surfaces in H^3 is equivalent to finding integrable Lax pairs of the form (2.11)-(2.12) and their solutions F. Then the surfaces are found by using the Sym-Bobenko type formulas (2.13) and (2.15) and (2.16). So to prove that the DPW recipe finds all of these types of surfaces, it is sufficient to prove that the DPW recipe produces all integrable Lax pairs of the form (2.11)-(2.12) and all their solutions F. The goal of this section is to show how this is done in [8].

3.1. **The loop groups.** Let C_r be the circle of radius $r \leq 1$ centered at the origin in $\boldsymbol{C}$.

Definition 1. For any $r \in (0,1] \subset \boldsymbol{R}$, we define the following loop groups:

(1) The twisted $sl(2,\boldsymbol{C})$ r-loop algebra is
$$\Lambda_r sl(2,\boldsymbol{C}) = \{A : C_r \to^{C^\infty} sl(2,\boldsymbol{C}) \mid A(-\lambda) = \sigma_3 A(\lambda)\sigma_3 \ \} \ .$$
(The condition $A(-\lambda) = \sigma_3 A(\lambda)\sigma_3$ is why we call the loop group "twisted".)

(2) The twisted $\mathrm{SL}(2,\boldsymbol{C})$ r-loop group is
$$\Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) = \{\phi : C_r \to^{C^\infty} \mathrm{SL}(2,\boldsymbol{C}) \mid \phi(-\lambda) = \sigma_3 \phi(\lambda)\sigma_3 \ \} \ .$$

(3) The twisted $\mathrm{SU}(2)$ r-loop group is
$$\begin{aligned}\Lambda_r \,\mathrm{SU}(2) = \{&F \in \Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) \mid F(1/\bar{\lambda})^* = (F(\lambda))^{-1} \ ,\\ &F = F(\lambda) \text{ extends holomorphically to } \lambda \text{ for } r < |\lambda| < r^{-1}\\ &\text{and continuously for } r \leq |\lambda| \leq r^{-1}\}\\ \cong \{&F : C_r \to^{C^\infty} \mathrm{SL}(2,\boldsymbol{C}) \mid F = F(\lambda) \text{ extends holomorphically to}\\ &\lambda \text{ for } r < |\lambda| \leq 1 \text{ and continously for } r \leq |\lambda| \leq 1 \text{ and } F|_{C_1} \in \mathrm{SU}(2) \ \} \ .\end{aligned}$$
When $r = 1$, we may abbreviate $\Lambda_1 \,\mathrm{SU}(2)$ to $\Lambda \,\mathrm{SU}(2)$. The condition in $\Lambda \,\mathrm{SU}(2)$ that F extends holomorphically is vacuous.

(4) The twisted plus r-loop group with $\boldsymbol{R}^+$ constant terms is
$$\begin{aligned}\Lambda_{+,r,\boldsymbol{R}^+} SL(2,\boldsymbol{C}) = \{&B \in \Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) \mid B \text{ extends holomorphically to } \lambda \text{ for } |\lambda| < r\\ &\text{and continuously for } |\lambda| \leq r \ , \text{ and } B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ with } \rho > 0 \ \} \ .\end{aligned}$$
When $r = 1$, we may abbreviate $\Lambda_{+,1,\boldsymbol{R}^+} SL(2,\boldsymbol{C})$ to $\Lambda_+ SL(2,\boldsymbol{C})$.

(5) The twisted plus r-loop group with general constant terms is
$$\begin{aligned}\Lambda_{+,r} SL(2,\boldsymbol{C}) = \{&B \in \Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) \mid B \text{ extends holomorphically to } \lambda \text{ for}\\ &|\lambda| < r \text{ and continously for } |\lambda| \leq r \ \} \ .\end{aligned}$$

(6) The twisted minus r-loop group with id constant terms is
$$\begin{aligned}\Lambda_{-,r,*} SL(2,\boldsymbol{C}) = \{&B \in \Lambda_r \,\mathrm{SL}(2,\boldsymbol{C}) \mid B \text{ extends holomorphically to } \lambda \text{ for}\\ &|\lambda| > r \text{ and continously for } |\lambda| \geq r \ , \text{ and } B|_{\lambda=\infty} = \mathrm{id} \ \} \ .\end{aligned}$$

3.2. **Iwasawa and Birkhoff splittings.** It is irrelevant how we topologize the loop algebra and loop groups, as long as the smooth loops are contained in the topology, since we will always be staying in the smooth category. However, to state the next two splitting lemmas, we must choose a topology. Let us choose the topology determined by the H^α norm for some $\alpha > 1/2$ (see [20]). With respect to this norm, all of the above smooth loops will have finite norm. (Loops with poles will probably not have finite norm.) We can then extend the above loop groups $\Lambda_r \,\mathrm{SL}(2,\boldsymbol{C})$, $\Lambda_r \,\mathrm{SU}(2)$, $\Lambda_{+,r,\boldsymbol{R}^+} SL(2,\boldsymbol{C})$, $\Lambda_{+,r} SL(2,\boldsymbol{C})$ and $\Lambda_{-,r,*} SL(2,\boldsymbol{C})$ to their completions with respect to the H^α norm. Then the notion of diffeomorphisms between these loops groups, and also the notion of smooth (resp. real-analytic, complex-analytic) dependence of the following splittings on z, makes sense.

Lemma 3.1. (Iwasawa decomposition) *For any $r \in (0,1]$, we have the following real-analytic diffeomorphism globally defined from $\Lambda_r \mathrm{SL}(2,\boldsymbol{C})$ to $\Lambda_r \mathrm{SU}(2) \times \Lambda_{r,+,\boldsymbol{R}^+} \mathrm{SL}(2,\boldsymbol{C})$: For any $\phi \in \Lambda_r SL(2,\boldsymbol{C})$, there exist unique $F \in \Lambda_r \mathrm{SU}(2)$ and $B \in \Lambda_{+,r} SL(2,\boldsymbol{C})$ so that*

$$\phi = FB \ .$$

We call this r-Iwasawa splitting of ϕ. We $r=1$, we may call it simply Iwasawa splitting. Because the diffeomorphism is real-analytic, we know that if ϕ depends real-analytically (resp. smoothly) on some parameter z, then F and B do as well.

Lemma 3.2. (Birkhoff decomposition) *For any $r \in (0,1]$, we have the following complex-analytic diffeomorphism defined from an open dense subset $\mathcal{U}$ of $\Lambda_r \mathrm{SL}(2,\boldsymbol{C})$ to $\Lambda_{-,r,*} \mathrm{SL}(2,\boldsymbol{C}) \times \Lambda_{+,r} \mathrm{SL}(2,\boldsymbol{C})$: For any $\phi \in \mathcal{U}$, there exist unique $B_- \in \Lambda_{-,r,*} \mathrm{SL}(2,\boldsymbol{C})$ and $B_+ \in \Lambda_{+,r} SL(2,\boldsymbol{C})$ so that*

$$\phi = B_- B_+ \ .$$

We call this r-Birkhoff splitting of ϕ. We $r=1$, we may call it simply Birkhoff splitting. Because the diffeomorphism is complex-analytic, we know that if ϕ depends complex-analytically (resp. real-analytically, smoothly) on some parameter z, then B_- and B_+ do as well.

From now on, whenever we apply these splitting results, it is sufficient to simply check that the loops we are splitting are smooth.

3.3. **The DPW method.** We now describe the DPW method. Let

$$\xi = A(z,\lambda) dz \ , \quad A(z,\lambda) \in \Lambda sl(2,\boldsymbol{C}) \ , \tag{3.1}$$

where $A := A(z,\lambda)$ is holomorphic in both z and λ for $z \in \Sigma$ and $\lambda \in \boldsymbol{C} \setminus \{0\}$. Furthermore, we assume the following:

$$\begin{pmatrix} A \text{ has a pole of order at most 1 at } \lambda = 0, \\ \text{and the upper-right entry of } A \text{ really does have a pole at } \lambda = 0. \end{pmatrix} \tag{3.2}$$

We call ξ a *holomorphic potential.*

In practice, when we wish to make specific examples of CMC surfaces, we will write A in the form

$$A = A_{-1}(z)\lambda^{-1} + A_0(z) + A_1(z)\lambda + A_2(z)\lambda^2 + \ldots \ ,$$

where the $A_j = A_j(z) \in M_{2\times 2}$ are holomorphic in $z \in \Sigma$ and do not depend on λ. By (3.2), we must choose A_{-1} so that its upper-right entry is never zero on Σ. Because $A \in \Lambda sl(2,\boldsymbol{C})$, A_j is off-diagonal (resp. diagonal) when j is odd (resp. even). Furthermore, all A_j are traceless. In fact, in all the example we later consider, only finitely many of the A_j will be nonzero.

Let ϕ be the solution to

$$d\phi = \phi\xi \ , \qquad \phi(z_*) = \mathrm{id}$$

for some base point $z_* \in \Sigma$. Then ϕ is holomorphic and

$$\phi \in \Lambda SL(2,\boldsymbol{C}) \ .$$

By Lemma 3.1 above, we can perform r-Iwasawa splitting, and write the result as

$$\phi = FB \ .$$

Remark. Because $F \in \Lambda_r \mathrm{SU}(2)$ and $\partial_{\bar\lambda} F = 0$ we have

$$((F(\lambda)^*)^{-1})_z = U(1/\bar\lambda)(F(\lambda)^*)^{-1} \ , \quad (F(1/\bar\lambda))_z = U(1/\bar\lambda)(F(1/\bar\lambda)) \ ,$$

and

$$((F(\lambda)^*)^{-1})_{\bar z} = V(1/\bar\lambda)(F(\lambda)^*)^{-1} \ , \quad (F(1/\bar\lambda))_{\bar z} = V(1/\bar\lambda)(F(1/\bar\lambda))$$

for all $\lambda \in \boldsymbol{C}$. So $(F(\lambda)^*)^{-1} = F(1/\bar\lambda)$ on S^1, and we have

$$(F(1/\bar\lambda)^*)^{-1} = F(\lambda) \cdot \mathcal{A}(\lambda) \ ,$$

where $\mathcal{A}(\lambda) = \mathrm{id}$ for all $\lambda \in S^1$ and $\mathcal{A}(\lambda)$ is independent of z and $\bar z$. Since $F(\lambda)$ and $(F(1/\bar\lambda)^*)^{-1}$ are both complex analytic in λ, $\mathcal{A}(\lambda)$ is also. This implies that $\mathcal{A}(\lambda) = \mathrm{id}$ for all $\lambda \in \boldsymbol{C}$, and hence we have the *reality condition*

$$F(1/\bar\lambda) = (F(\lambda)^*)^{-1} \ . \tag{3.3}$$

Proposition 3.3. *Up to a conformal change of the coordinate z, F is a solution to a Lax pair of the form* (2.11)-(2.12).

Proof. First, we define holomorphic $a_{-1} = a_{-1}(z)$ and $b_{-1} = b_{-1}(z)$, and real $\rho = \rho(z, \bar z) \in \boldsymbol{R}^+$ by

$$(\lambda A)|_{\lambda=0} = \begin{pmatrix} 0 & a_{-1} \\ b_{-1} & 0 \end{pmatrix} \text{ and } B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \ .$$

If we use the new conformal coordinate

$$w = \int_{z_0}^{z} \frac{1}{2a_{-1}} dz \ ,$$

which is really a new coordinate because a_{-1} is never zero, we have that b_{-1} changes to $Q/2$, where we define Q to be $Q := b_{-1}/a_{-1}$, and a_{-1} changes to $1/2$. Using $\Phi = F \cdot B$, we can rewrite $U(\lambda)$ as the following:

$$U(\lambda) = F^{-1} F_z = BAB^{-1} - B^{-1} B_z = \sum_{k=-1}^{\infty} U_k \lambda^k \ ,$$

$$V(\lambda) = F^{-1} F_{\bar z} = -B^{-1} B_{\bar z} = \sum_{k=0}^{\infty} V_k \lambda^k \ ,$$

where

$$U_{-1} = \frac{1}{2} \begin{pmatrix} 0 & \rho^2 \\ Q\rho^{-2} & 0 \end{pmatrix} \ , \quad V_0 = -\frac{\rho_{\bar z}}{\rho} \sigma_3 \ .$$

One can then prove that

$$\overline{U(1/\bar\lambda)} = \sigma_2 V(\lambda) \sigma_2 \ ,$$

and therefore $U(\lambda) = U_{-1}\lambda^{-1} + U_0$ and $V(\lambda) = V_0 + V_1 \lambda$, where $U_0 = -\bar V_0^t$ and $V_1 = -\bar U_{-1}^t$, and the proof is done. □

Remark. When doing Iwasawa splitting, the condition that $\rho \in \boldsymbol{R}^+$ makes the splitting $\phi = FB$ unique. But actually it is OK to more generally allow that $\rho \in \boldsymbol{C}$ is not real. Then although the splitting becomes non-unique – let us call it $\phi = \tilde F \tilde B$ for some $\rho = re^{i\theta} \in \boldsymbol{C}$ – it follows that

$$B = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tilde B \ , \quad F = \tilde F \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \ ,$$

since the matrix $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ is in $SU(2)$. Thus F and $\tilde F$ are not the same, but using either one in equation (2.13) results in the same immersion f.

Proposition 3.4. *For any solution $F \in \Lambda\,\mathrm{SU}(2)$, defined (need to say?) for all $z \in \Sigma$ and all $\lambda \in \boldsymbol{C} \setminus \{0\}$ with $F(z_*) = \mathrm{id}$, to a Lax pair of type* (2.11)-(2.12), *there exists a holomorphic potential $\xi = Adz$ with A as in* (3.1) *and a solution $\phi \in \Lambda\,\mathrm{SL}(2,\boldsymbol{C})$ of $d\phi = \phi\xi$ so that ϕ Iwasawa splits into $\phi = FB$ for some $B \in \Lambda_+ SL(2,\boldsymbol{C})$. ξ is called the* holomorphic potential *of the CMC surface resulting from F.*

Proof. The wish to find a $B \in \Lambda_+ SL(2,\boldsymbol{C})$ so that $(FB)_{\bar z} = 0$, and then define $\phi := FB$ and then show $A := \phi^{-1}\phi_z$ is of the form in (3.1). If $(FB)_{\bar z} = F_{\bar z}B + FB_{\bar z} = FVB + FB_{\bar z} = 0$ with V as in (2.12), then $B_{\bar z} = -VB$, and we need to solve this for B. The existence of such a B follows from the dell bar problem.

Also, because V satisfies the twistedness condition, if B satisfies the twistedness condition at one value of z for all λ, then B satisfies the twistedness condition everywhere. Therefore,

$$B \in \Lambda_+ SL(2,\boldsymbol{C})\ .$$

Since the equation for B is linear, B has no singularities for any z and λ, and so B^{-1} also has no singularities for any z and λ.

Furthermore, F and F^{-1} always exist without singularities, because F solves (2.11)-(2.12) with nonsingular initial condition. In particular, defining

$$\phi := FB\ ,$$

ϕ and ϕ^{-1} are both holomorphic without singularities.

Define A by $A = \phi^{-1}\phi_z$. Then A is holomorphic in z and λ for $z \in \Sigma$ and $\lambda \in \boldsymbol{C} \setminus \{0\}$. Also,

$$\lambda A = \lambda\phi^{-1}\phi_z = \lambda B^{-1}UB + \lambda B^{-1}B_z$$

is bounded at $\lambda = 0$, since $B \in \Lambda_+\,\mathrm{SL}(2,\boldsymbol{C})$. Therefore A has at most a pole of order 1 at $\lambda = 0$, and so A is of the desired form (3.1). □

We have established the converse of DPW for a holomorphic potential.

Remark. In the above proof, if

$$B^{-1} \to B^{-1}\begin{pmatrix} e^{i\theta(z,\bar z)} & 0 \\ 0 & e^{-i\theta(z,\bar z)} \end{pmatrix}$$

for the right choice of $\theta(z,\bar z)$, we can change B so that B_0 has positive reals on the diagonal. Then

$$F \to F\begin{pmatrix} e^{i\theta(z,\bar z)} & 0 \\ 0 & e^{-i\theta(z,\bar z)} \end{pmatrix}\ ,$$

but this does not change the surface (see the Sym-Bobenko formula (2.13)).

3.4. **The meaning of dressing and gauging.** Given a solution ϕ to $d\phi = \phi\xi$, if we define

$$\hat\phi = h_+(\lambda)\cdot\phi\cdot p_+(z,\bar z,\lambda)\ , \qquad h_+, p_+ \in \Lambda_+\,\mathrm{SL}(2,\boldsymbol{C})\ ,$$

then the multiplication on the left by h_+ is a dressing, and the multiplication on the right by p_+ is a gauging. The matrix h_+ cannot depend on z. The matrix p_+ can depend on z, but must have trivial monodromy about all loops in the z-domain.

Note that $\hat\phi$ satisfies $d\hat\phi = \hat\phi\hat\xi$, where

$$\hat\xi = p_+^{-1}\xi p_+ + p_+^{-1}dp_+\ .$$

Hence,

> the dressing h_+ does not change the potential ξ, and changes only the resulting surface.

Furthermore, if we look at the Iwasawa splittings $\phi = FB$ and $\hat{\phi} = \hat{F}\hat{B}$, then the change $F \to \hat{F}$ is affected only by h_+, and is independent of p_+, hence

> the gauging p_+ does not change the surface, and changes only the potential ξ.

To see how the surface is changed by h_+, one must Iwasawa split h_+F into $h_+F = \tilde{F}\tilde{B}$, and then $\hat{F}$ equals $\tilde{F}$, so the change in the frame is not trivial to understand, hence the change in the surface is also not trivial to understand.

However, it is easier to understand how the monodromy matrices of ϕ and $\hat{\phi}$ are related by h_+, and this is often just the information we need, because we are interested in getting the monodromy matrices into SU(2) so we can solve period problems. Define M and $\hat{M}$ by

$$\phi \to M\phi\ , \qquad \hat{\phi} \to \hat{M}\hat{\phi}$$

as one travels about some loop in the z-domain. Then it is simple (i.e. Iwasawa splitting is not required) to check that

$$\hat{M} = h_+ M h_+^{-1}\ .$$

3.5. **Period problems in S^3 and H^3.** In the case of $\boldsymbol{R}^3$, we have a six real dimensional period problem for each homology class of loops, as in [15]. If M is defined so that $\phi \to M \cdot \phi$ about a loop on the Riemann surface (with local coordinate z), then the Sym-Bobenko formula implies that the immersion changes as

$$f \to \left[M f M^{-1} - i\lambda(\partial_\lambda M) M^{-1} \right]_{\lambda=1}$$

as one travels about the loop. M is independent of z, but not of λ. Supposing that we already know $M \in SU(2)$ for all $\lambda \in S^1$, then for the surface to be well defined about the loop we need to know that

$$M|_{\lambda=1} = \pm id \quad \text{and} \quad \partial_\lambda M|_{\lambda=1} = 0\ , \tag{3.4}$$

that is, we need to get $(M, \partial_\lambda M)|_{\lambda=1}$ to be the identity element (up to sign) in $SU(2) \times su(2)$. Since the dimension of the space $SU(2) \times su(2)$ is six, the period problem is six dimensional.

For the cases of S^3 and H^3, we check here that again the period problem is six dimensional.

H^3 case, $H > 1$: About a loop we have $F \to M \cdot F$, and assume that we already know M is unitary on S^1 (i.e. $(M(\lambda)^*)^{-1} = M(1/\bar{\lambda})$ for all $\lambda \in \boldsymbol{C} \setminus \{0\}$). About the loop, the immersion changes as

$$f \to \left[M f \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \overline{M}^{-1} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right]_{\lambda = e^{q/2}}\ .$$

So for the surface to be well defined about the loop, we need

$$M|_{\lambda=e^{q/2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm id\ , \tag{3.5}$$

that is, we need the identity element (up to sign) in $SL(2,\boldsymbol{C})$. (Note that even though M is unitary on S^1, we can only consider the problem in $SL(2,\boldsymbol{C})$, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\boldsymbol{C}) \ ,$$

because $\lambda = e^{q/2}$ does not lie on S^1.) Since $SL(2,\boldsymbol{C})$ is six dimensional, so is the period problem.

S^3 **case:** Again assume M_1, M_2, defined by $F_j \to M_j \cdot F_j$ as we travel about the loop, are unitary on S^1. The pair $(\lambda_1, \lambda_2) = (1, e^{2i\psi})$ implies

$$f = F_1 \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} F_2^{-1} \ .$$

So when we travel about the loop, we have

$$f \to M_1 f M_2^{-1} \ .$$

To close the surface about this loop, we need

$$M_1 = M_2 = \pm \mathrm{id} \ . \tag{3.6}$$

Note that M_j are in SU(2), since $|\lambda_j| = 1$. So we need the identity element (up to sign) in $SU(2) \times SU(2)$. As $SU(2) \times SU(2)$ is six dimensional, so is the period problem.

4. Surfaces of Revolution

4.1. **Cylinders via DPW.** Define

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \frac{dz}{z} \ ,$$

for the complex variable $z \in \boldsymbol{C}$ and $\lambda \in S^1$ and $a \in \boldsymbol{R}$. One solution to $d\phi = \phi\xi$ is

$$\phi = \exp\left(\log z \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \lambda^{-1} \right) = \begin{pmatrix} \cosh(a\lambda^{-1}\log z) & \sinh(a\lambda^{-1}\log z) \\ \sinh(a\lambda^{-1}\log z) & \cosh(a\lambda^{-1}\log z) \end{pmatrix} \ ,$$

which has Iwasawa splitting

$$\phi = FB \ , \quad \text{where } B = \exp\left(\lambda \log \bar{z} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \right) \text{ and}$$

$$F = \exp\left((\lambda^{-1}\log z - \lambda \log \bar{z}) \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \right) \ .$$

When one travels once counterclockwise about the origin in the z-plane, F changes to $M \cdot F$, where M is a matrix depending only on λ, and

$$M = \begin{pmatrix} \cosh(2\pi i a(\lambda^{-1} + \lambda)) & \sinh(2\pi i a(\lambda^{-1} + \lambda)) \\ \sinh(2\pi i a(\lambda^{-1} + \lambda)) & \cosh(2\pi i a(\lambda^{-1} + \lambda)) \end{pmatrix} \in \Lambda\,\mathrm{SU}(2) \ .$$

Since we will always have $\lambda \in \boldsymbol{R}$ or $\lambda \in S^1$, we have $\lambda + \lambda^{-1} \in \boldsymbol{R}$. Hence

$$M = \begin{pmatrix} \cos(2\pi a(\lambda^{-1} + \lambda)) & i\sin(2\pi a(\lambda^{-1} + \lambda)) \\ i\sin(2\pi a(\lambda^{-1} + \lambda)) & \cos(2\pi a(\lambda^{-1} + \lambda)) \end{pmatrix} \ .$$

When $\mathcal{M}^3 = \boldsymbol{R}^3$, we choose $a = 1/4$. When $\mathcal{M}^3 = H^3$, we choose $\lambda = e^{q/2}$ for $q \in \boldsymbol{R}^+$ and $a = 1/(4\cosh(q/2))$, so $\lambda > 1$ and the resulting surface has mean curvature $H = \coth q > 1$. When $\mathcal{M}^3 = S^3$, we choose $\lambda_1 = e^{i\gamma}$ and $\lambda_2 = e^{-i\gamma}$ for $\gamma \in (0, \pi/4]$ and $a = 1/(4\cos\gamma)$, so the resulting surface has mean curvature $H = \cot(2\gamma)$.

In each of these three space forms, the conditions (3.4), (3.5) and (3.6) are satisfied, respectively. Hence in all three cases we have produced surfaces that are homeomorphically cylinders.

Inserting F into equations (2.13), (2.15), and (2.16), one can explicitly compute the parametrizations for the surfaces and see that cylinders are produced. In the case of S^3, the cylinder wraps around onto itself to become of torus, since the geodesic lines in S^3 are loops.

Remark. Viewing cylinders as members of a larger family of Delaunay surfaces, we use a slightly more complicated potential ξ in section 4.2 that will also produce cylinders for certain values of its parameters.

4.2. **Delaunay surfaces via DPW.** Delaunay surfaces via DPW in $\boldsymbol{R}^3$ are described in detail in [17].

Define

$$\xi = D\frac{dz}{z} , \qquad \text{where } D = \begin{pmatrix} r & s\lambda^{-1} + t\lambda \\ s\lambda + t\lambda^{-1} & -r \end{pmatrix} ,$$

with $r, s, t \in \boldsymbol{R}$.

We remark that more generally one can allow s and t to be complex numbers, and assume $st \in \boldsymbol{R}$ (and $|s+\bar{t}|^2 + r^2 = 1/4$ in the $\boldsymbol{R}^3$ case), then gauging

$$F \to F \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

for some θ makes s, t real. The bulge and neck radii will be $(1/(2H))(1 \pm \sqrt{1-16st})$.

One solution of $d\phi = \phi\xi$ is

$$\phi = \exp(\ln z \cdot D) .$$

ϕ can be split (this is *not* Iwasawa splitting) in the following way:

$$\phi = F_1 B_1 , \quad F_1 = \exp(i\theta D) , \quad B_1 = \exp(\ln\rho \cdot D) ,$$

where $z = \rho e^{i\theta}$. Note that $F_1 \in \Lambda\,\mathrm{SU}(2)$.

Since $D^2 = X^2\,\mathrm{id}$, where $X = \sqrt{r^2 + (s+t)^2 + st(\lambda - \lambda^{-1})^2}$ we see that

$$F_1 = \begin{pmatrix} \cos(\theta X) + irX^{-1}\sin(\theta X) & iX^{-1}\sin(\theta X)(s\lambda^{-1} + t\lambda) \\ iX^{-1}\sin(\theta X)(s\lambda + t\lambda^{-1}) & \cos(\theta X) - irX^{-1}\sin(\theta X) \end{pmatrix} , \tag{4.1}$$

$$B_1 = \begin{pmatrix} \cosh(\ln\rho \cdot X) + rX^{-1}\sinh(\ln\rho \cdot X) & X^{-1}\sinh(\ln\rho \cdot X)(s\lambda^{-1} + t\lambda) \\ X^{-1}\sinh(\ln\rho \cdot X)(s\lambda + t\lambda^{-1}) & \cosh(\ln\rho \cdot X) - rX^{-1}\sinh(\ln\rho \cdot X) \end{pmatrix} .$$

Note that F_1 and B_1 are both even in λ along the diagonal and odd in λ on the off-diagonal. We can now do Iwasawa splitting on B_1, i.e. $B_1 = F_2 \cdot B$, where $F_2 \in \Lambda SU(2)$ and $B \in \Lambda_+ SL(2, \boldsymbol{C})$. We define $F = F_1 \cdot F_2$. Thus $\phi = FB$ is the Iwasawa splitting of ϕ.

Because, for each fixed λ, F_2 and B depend only on $|z| = \rho$ and F_1 depends only on θ, we have that, under the rotation of the domain

$$z \to R_{\theta_0}(z) = e^{i\theta_0} z ,$$

the following transformations occur:

$$F \to M_{\theta_0} F , \qquad B \to B ,$$

where

$$M_{\theta_0} = e^{i\theta_0 D} .$$

Note that M_{θ_0} in (4.1) depends on λ, and $M_{\theta_0} \in \Lambda\,\mathrm{SU}(2)$. In fact, M_{θ_0} is explicitly of the form F_1 in (4.1) evaluated at $\theta = \theta_0$.

Now we consider the period closing conditions in each of the three space forms:

- When $\mathcal{M}^3 = \boldsymbol{R}^3$, we want $M_{2\pi}$ to satisfy (3.4), so that the surface will close under a loop about the origin in the z-plane and become homeomorphically a cylinder. And (3.4) is satisfied if
$$r^2 + (s+t)^2 = 1/4\,,$$
so we impose this condition when $\mathcal{M}^3 = \boldsymbol{R}^3$.
- When $\mathcal{M}^3 = H^3$, we want $M_{2\pi}$ to satisfy (3.5), so that the surface will close under a loop about the origin in the z-plane. With $q \in \boldsymbol{R}^+$, (3.5) is satisfied if
$$r^2 + (s+t)^2 + 4st\sinh^2(\frac{q}{2}) = 1/4\,,$$
so we impose this when $\mathcal{M}^3 = H^3$.
- When $\mathcal{M}^3 = S^3$, we want $M_{2\pi}$ to satisfy (3.6), to make the surface close. With $\lambda_1 = e^{i\gamma}$ and $\lambda_2 = e^{-i\gamma}$, (3.6) is satisfied if
$$r^2 + (s+t)^2 - 4st\sin^2(\gamma) = 1/4\,,$$
so we impose this when $\mathcal{M}^3 = S^3$.

In the case of $\boldsymbol{R}^3$, under the mapping $z \to R_{\theta_0}(z)$, we have that f as in (2.13) changes as

$$f \to M_{\theta_0} f M_{\theta_0}^{-1} - i(\partial_\lambda M_{\theta_0})|_{\lambda=1} M_{\theta_0}^{-1}\,. \tag{4.2}$$

One can check that Equation (4.2) represents a rotation of angle θ_0 about the line

$$\{x \cdot (-s-t, 0, r) + 2(s-t)\cdot(2r, 0, 2s+2t) \,|\, x \in \boldsymbol{R}\}\,.$$

Therefore f is a surface of revolution, and hence a Delaunay surface. Which Delaunay surface one gets depends on the choice of r, s, t. An unduloid is produced when $st > 0$. If we allow s or t to be nonpositive, then a nodoid is produced when $st < 0$, and for the limiting singular case of a chain of spheres, $st = 0$. A cylinder is produced when $s = t$.

In the case of H^3, under the mapping $z \to R_{\theta_0}(z)$, we have that f as in (2.16) changes as

$$f \to M_{\theta_0} f \sigma_2 \overline{M_{\theta_0}^{-1}} \sigma_2\,. \tag{4.3}$$

One can check that Equation (4.3) represents a rotation of angle θ_0 about the geodesic line

$$\{(x_0, x_1, 0, x_3) \in \boldsymbol{R}^{3,1} \mid (e^q - 1)(s-t)x_0 - re^{q/2}x_1 + (e^q+1)(s+t)x_3 = 0\} \cap H^3\,.$$

Therefore f is a surface of revolution, and hence a Delaunay surface in H^3.

Remark. We still need to find the axis in the S^3 case. We also need to give the weights of these Delaunay surfaces in all three space forms. Knowing the weights will be important for describing trinoids.

5. Bubbletons

5.1. **Bubbletons via DPW.** Let $\mathcal{R}$ be the Riemann surface $S^2\backslash\{p_1,p_2\}$ with the standard holomorphic structure. Using stereographic projection, we can denote $\mathcal{R} = \boldsymbol{C}\cup\{\infty\}\backslash\{p_1,p_2\}$. And using a Moebius transformation, we can transform $\mathcal{R}$ to $\boldsymbol{C}^* = \boldsymbol{C}\backslash\{0\}$. Stereographic projection and Moebius transformations preserve the holomorphic structure of the Riemann surface. So we need only consider $\mathcal{R} = \boldsymbol{C}^* = \boldsymbol{C}\backslash\{0\}$.

Let $\phi(z,\lambda)$ be a solution of $d\phi = \phi\xi$ with some initial condition $\phi(z_*,\lambda)$ at $z = z_*$ and let $\phi = F\cdot B$ be the r-Iwasawa splitting of ϕ, where $\xi = A(z,\lambda)dz$ and $A(z,\lambda) \in \Lambda_r sl(2,\boldsymbol{C})$ for some $r \in (0,1]$. Let f be as in the Sym-Bobenko formula (2.13) or (2.15) or (2.16), respectively, made from the extended frame F. We assume that the monodromy M_ϕ of ϕ (associated to a counterclockwise loop around $z = 0$) is in $\Lambda_r SU(2)$ and M_ϕ satisfies one of the closing conditions (3.4) or (3.5) or (3.6), respectively. Thus f is well-defined on $\mathcal{R}$.

Remark. The first assumption that M_ϕ is in $\Lambda_r SU(2)$ is not actually a restriction, because if $\mathrm{tr}M_\phi$ is in $(-1,1)$ or M_ϕ equals to $\pm id$, then we can change the initial condition $\phi(z_*,\lambda)$ to $C(\lambda)\cdot\phi(z_*,\lambda)$ for some $C(\lambda)$ so that M_ϕ changes to $C(\lambda)M_\phi C^{-1}(\lambda)$ and $C(\lambda)M_\phi C^{-1}(\lambda)$ is in $\Lambda_r SU(2)$.

Consider the dressing $\phi\to\tilde{\phi} := h\cdot\phi$, where h is the matrix

$$h = \begin{pmatrix} \sqrt{\frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2}} & 0 \\ 0 & \sqrt{\frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2}} \end{pmatrix},\ \alpha\in\boldsymbol{C}^*.$$

Let $\tilde{\phi} = \tilde{F}\cdot\tilde{B}$ be the r-Iwasawa splitting of $\tilde{\phi}$ and let $\tilde{f}$ be the Sym-Bobenko formula (2.13) or (2.15) or (2.16), respectively, made from the extended frame $\tilde{F}$. Note that if $|\alpha| < r$ or $r^{-1} < |\alpha|$, then $h\in\Lambda_r SU(2)$. So the surface $\tilde{f}$ differs from f by only a rigid motion. Therefore we assume $r < |\alpha| < 1$.

Lemma 5.1. *If $hM_\phi h^{-1}$ in $\Lambda_r SU(2)$, then $\tilde{F}$ changes to $(hM_\phi h^{-1})\cdot\tilde{F}$ when one travels a counterclockwise loop around $z = 0$. Hence the monodromy of $\tilde{F}$ is $hM_\phi h^{-1}$.*

Proof. When one travels a counterclockwise loop around $z = 0$, $\tilde{\phi}$ changes as follows

$$\tilde{\phi}\to hM_\phi h^{-1}\tilde{\phi}$$

The condition $hM_\phi h^{-1}\in\Lambda_r SU(2)$ implies that r-Iwasawa splitting of $hM_\phi h^{-1}\tilde{\phi}$ is $(hM_\phi h^{-1}\tilde{F})\tilde{B}$, i.e $hM_\phi h^{-1}\tilde{F}\in\Lambda SU(2)$ and $\tilde{B}\in\Lambda_{r+}SL(2,\boldsymbol{C})$. Thus $\tilde{F}$ changes $hM_\phi h^{-1}\tilde{F}$ when one travels a counterclockwise loop around $z = 0$. □

Noting the previous lemma, we define the bubbleton surfaces.

Definition 2. Let $f,\tilde{f}:\mathcal{R}\longrightarrow\boldsymbol{R}^3$ or H^3 or S^3, be CMC immersions derived from the above solutions ϕ and $\tilde{\phi}$. Then $\tilde{f}$ is a bubbleton surface of f if $hM_\phi h^{-1}\in\Lambda_r SU(2)$.

Lemma 5.2. *The bubbleton $\tilde{f}$ satisfies the closing condition: that is, it is well-defined on $\mathcal{R}$.*

Proof. In the $\boldsymbol{R}^3$ case, we show that since $M_\phi|_{\lambda=1} = \pm id$ and $\partial_\lambda M_\phi|_{\lambda=1} = 0$ are satisfied, thus $(hM_\phi h^{-1})|_{\lambda=1} = \pm id$ and $\partial_\lambda(hM_\phi h^{-1})|_{\lambda=1} = 0$ are also satisfied. This follows from the following computations:

$$(hM_\phi h^{-1})|_{\lambda=1} = h|_{\lambda=1} M_\phi|_{\lambda=1} h^{-1}|_{\lambda=1} = \pm id \ ,$$
$$\partial_\lambda(hM_\phi h^{-1})|_{\lambda=1} =$$
$$((\partial_\lambda h) M_\phi h^{-1})|_{\lambda=1} + (h(\partial_\lambda M_\phi) h^{-1})|_{\lambda=1} + (hM_\phi(-h^{-1}(\partial_\lambda h)h^{-1}))|_{\lambda=1} = 0 \quad .$$

The H^3 and S^3 cases are similar, in fact they are even simpler, because no derivatives with respect to λ are involved. ☐

Lemma 5.3. *$hM_\phi h^{-1}$ is in $\Lambda_r SU(2)$ if and only if M_ϕ is an upper triangular matrix at $\lambda = \pm\alpha$ and a lower triangular matrix at $\lambda = \pm\bar{\alpha}^{-1}$.*

Proof. Let m_{ij} be the entries of M_ϕ. We have

$$hM_\phi h^{-1} = \begin{pmatrix} m_{11} & \frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2} m_{12} \\ \frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2} m_{21} & m_{22} \end{pmatrix} .$$

Thus $hM_\phi h^{-1}$ is in $\Lambda_r SU(2)$ iff $\frac{1-\bar{\alpha}^2\lambda^2}{\lambda^2-\alpha^2} m_{12}(\lambda)$ and $\frac{\lambda^2-\alpha^2}{1-\bar{\alpha}^2\lambda^2} m_{21}(\lambda)$ are holomophic on $r < |\lambda| < r^{-1}$. This happens iff $\frac{m_{12}(\lambda)}{\lambda^2-\alpha^2}$ and $\frac{m_{21}(\lambda)}{1-\bar{\alpha}^2\lambda^2}$ are bounded on $r \le |\lambda| \le r^{-1}$. And this happens iff M_ϕ is an upper triangular matrix at $\lambda = \pm\alpha$ and a lower triangular matrix at $\lambda = \pm\bar{\alpha}^{-1}$. ☐

Theorem 5.4. *There exist cylinder bubbleton and Delaunay bubbleton surfaces for all three space forms.*

Proof. First we consider the case that f is a cylinder, with ϕ, ξ, M_ϕ as in Section 4.1. So the monodromy matrix is:

$$M_\phi = \begin{pmatrix} \cosh(2\pi a(\lambda+\lambda^{-1})) & \sinh(2\pi a(\lambda+\lambda^{-1})) \\ \sinh(2\pi a(\lambda+\lambda^{-1})) & \cosh(2\pi a(\lambda+\lambda^{-1})) \end{pmatrix} ,$$

where

$$\begin{cases} \boldsymbol{R}^3 \text{ case: } a = 1/4 \\ H^3 \text{ case: } a = 1/(4\cosh(q/2)) \\ S^3 \text{ case: } a = 1/(4\cos(\gamma)) \end{cases} .$$

Clearly M_ϕ is in $\Lambda_r SU(2)$ for all $r \in (0,1]$ and satisfies the closing conditions. We take $\alpha = \delta - \sqrt{\delta^2-1}$ with $\delta = \frac{k}{4a}$ $\ (k \ge 2$ and $k \in N)$, and we can immediately compute $M_\phi|_{\lambda=\pm\alpha,\pm\alpha^{-1}} = -id$. We can choose r so that α satisfies $r < |\alpha| < 1$. Thus Lemma 5.3, Lemma 5.2 and Definition 2 imply existence of the bubbleton surface of the cylinder.

Now we consider the case that f is a delaunay surface, with ϕ, ξ, M_ϕ as in Section 4.2. In this case the monodromy matrix is:

$$M_\phi = \begin{pmatrix} \cos(2\pi X) + irX^{-1}\sin(2\pi X) & iX^{-1}\sin(2\pi X)(s\lambda^{-1}+t\lambda) \\ iX^{-1}\sin(2\pi X)(s\lambda+t\lambda^{-1}) & \cos(2\pi X) - irX^{-1}\sin(2\pi X) \end{pmatrix} ,$$

where

$$X = \sqrt{r^2+(s+t)^2+st(\lambda-\lambda^{-1})^2},\ r^2+(s+t)^2+a = \frac{1}{4}$$

, and

$$\begin{cases} \boldsymbol{R}^3 \text{case:} a = 0 \\ H^3 \text{ case: } a = 4st\cosh^2(q/2) \\ S^3 \text{ case: } a = -4st\sin^2(\gamma) \end{cases} .$$

Again clearly M_ϕ is in $\Lambda_r SU(2)$ for all $r \in (0,1]$ and satisfies the closing conditions. We take $\alpha = \frac{\sqrt{\delta+4}-\sqrt{\delta}}{2}$ with $\delta = \frac{1}{st}(\frac{k^2-1}{4}+a)$ $(k \geq 2$ and $k \in N)$, and we can immediately compute $M_\phi|_{\lambda=\pm\alpha,\pm\alpha^{-1}} = -id$. We can choose r so that α satisfies $r < |\alpha| < 1$. Thus again Lemma 5.3, Lemma 5.2 and Definition 2 imply existence of the bubbleton surface of the delaunay surface. □

Remark. We consider the dressing matrixes $h_1, ..., h_n$:

$$h_i = \begin{pmatrix} \sqrt{\frac{1-\bar{\alpha_i}^2\lambda^2}{\lambda^2-\alpha_i^2}} & 0 \\ 0 & \sqrt{\frac{\lambda^2-\alpha_i^2}{1-\bar{\alpha_i}^2\lambda^2}} \end{pmatrix}, \quad \text{where } \alpha_i \in \boldsymbol{C}^* \text{ and } h_i \neq h_j \text{ if } i \neq j \ .$$

We put $H := h_1 \cdots h_n$ and we consider the dressing $\phi \to \tilde{\Phi} := H \cdot \phi$. We can do the same arguments in this section using H instead of the previous h, and then we see that there exist multibubbletons with cylindrical and Delaunay ends.

5.2. **Computing the change of frame for the simple type dressing.** Now we do the story of the Bäcklund transformation in the sense of Terng and Uhlenbeck (see [25]). This will lead to explicit parametrization of the cylinder bubbletons in all three space forms.

Let ϕ be a solution of $d\phi = \phi\xi$ with the some initial condition $\phi(z_*, \lambda)$ and let $\phi = F \cdot B$ be the r-Iwasaswa splitting. In this section, the situation and the assumptions are the same as in Section 5. We consider $\boldsymbol{C}^2$ with inner product $\langle , \rangle$ and e_1, e_2 forming the orthonormal basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

of $\boldsymbol{C}^2$. We define two subspace V_1, V_2 spaned by v_1, v_2.:

$$V_1 := \{a \cdot v_1 | v_1 = \begin{pmatrix} \bar{A} \\ \lambda^{-1}\bar{\alpha}^{-1}\bar{B} \end{pmatrix}, a \in \boldsymbol{C}\}, \qquad V_2 := \{a \cdot v_2 | v_2 = \begin{pmatrix} -\lambda\alpha^{-1}B \\ A \end{pmatrix}, a \in \boldsymbol{C}\}$$

where

$$F|_{\lambda=\alpha} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

We define projections $\pi_1, \pi_2, \tilde{\pi}_1, \tilde{\pi}_2$ and linear combinations $h, \tilde{h}$ of these projections.

$$\begin{cases} \pi_1 & := \text{ orthogonal projection to } e_1 \\ \pi_2 & := \text{ orthogonal projection to } e_2 \\ h & := f^{-1/2}\pi_1 + f^{1/2}\pi_2 \end{cases} \qquad \begin{cases} \tilde{\pi}_1 & := \text{projection to } V_1 \text{ parallel to } V_2 \\ \tilde{\pi}_2 & := \text{projection to } V_2 \text{ parallel to } V_1 \\ \tilde{h} & := f^{-1/2}\tilde{\pi}_1 + f^{1/2}\tilde{\pi}_2 \end{cases}$$

where

$$f = \frac{\lambda^2 - \alpha^2}{1 - \bar{\alpha}^2\lambda^2}.$$

Note that in general $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are non-orthogonal projections.

The following lemma is obvious:

Lemma 5.5.

$$\pi_1 \circ \pi_1 = \pi_1 \ , \pi_1 \circ \pi_2 = 0 \ , \pi_2 \circ \pi_1 = 0 \ , \pi_2 \circ \pi_2 = \pi_2 \ .$$

Lemma 5.6.

$$h^{-1} = f^{1/2}\pi_1 + f^{-1/2}\pi_2 \ , \qquad \tilde{h}^{-1} = f^{1/2}\tilde{\pi}_1 + f^{-1/2}\tilde{\pi}_2 \quad .$$

Proof.

$$\begin{aligned} h \circ h^{-1} &= (f^{-1/2}\pi_1 + f^{1/2}\pi_2) \circ (f^{1/2}\pi_1 + f^{-1/2}\pi_2) \\ &= \pi_1 \circ \pi_1 + f^{-1}\pi_1 \circ \pi_2 + f\pi_2 \circ \pi_1 + \pi_2 \circ \pi_2 = \pi_1 + \pi_2 = id\,, \end{aligned}$$

by Lemma 5.5. Similarly $h^{-1} \cdot h = id$.

$$\begin{aligned} \tilde{h} \circ \tilde{h}^{-1} &= (f^{-1/2}\tilde{\pi}_1 + f^{1/2}\tilde{\pi}_2) \circ (f^{1/2}\tilde{\pi}_1 + f^{-1/2}\tilde{\pi}_2) \\ &= \tilde{\pi}_1 \circ \tilde{\pi}_1 + f^{-1}\tilde{\pi}_1 \circ \tilde{\pi}_2 + f\tilde{\pi}_2 \circ \tilde{\pi}_1 + \tilde{\pi}_2 \circ \tilde{\pi}_2 = \tilde{\pi}_1 + \tilde{\pi}_2 = id\,, \end{aligned}$$

by Lemma 5.5. Similarly $\tilde{h}^{-1} \cdot \tilde{h} = id$. □

Lemma 5.7. *In terms of the basis e_1, e_2, we can write $\pi_j, \tilde{\pi}_j$ $(j = 1, 2)$ in the following matrix forms:*

$$\pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \pi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

$$\tilde{\pi}_1 = \frac{1}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} |A|^2 & \lambda\alpha^{-1}\bar{A}B \\ \lambda^{-1}\bar{\alpha}^{-1}A\bar{B} & |\alpha|^{-2}|B|^2 \end{pmatrix} \ ,$$

$$\tilde{\pi}_2 = \frac{1}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} |\alpha|^{-2}|B|^2 & -\lambda\alpha^{-1}\bar{A}B \\ -\lambda^{-1}\bar{\alpha}^{-1}A\bar{B} & |A|^2 \end{pmatrix} \ .$$

Proof. Consider $\pi_1, \pi_2, \tilde{\pi}_1, \tilde{\pi}_2$ as the above matrices. Then

$$\text{for all } \ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \boldsymbol{C}^2,$$

$$\pi_1 \cdot x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \ \text{ and } \ \pi_2 \cdot x = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

These imply that π_j is orthognoal projection to e_j for $j = 1,\ 2$. Similarly, we have

$$\tilde{\pi}_1 \cdot x = \frac{Ax_1 + \lambda\alpha^{-1}Bx_2}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} \bar{A} \\ \lambda^{-1}\bar{\alpha}^{-1}\bar{B} \end{pmatrix} \ , \quad \tilde{\pi}_2 \cdot x = \frac{-\lambda^{-1}\bar{\alpha}^{-1}\bar{B}x_1 + \bar{A}x_2}{|A|^2 + |\alpha|^{-2}|B|^2} \begin{pmatrix} -\lambda\alpha^{-1}B \\ A \end{pmatrix} \ .$$

Thus $\tilde{\pi}_1 \cdot v_1 = v_1$, $\forall v_1 \in V_1$ and $\tilde{\pi}_1 \cdot V_2 = 0$, and $\tilde{\pi}_2 \cdot v_2 = v_2$, $\forall v_2 \in V_2$ and $\tilde{\pi}_2 \cdot V_1 = 0$. These imply that $\tilde{\pi}_1$ is projection to V_1 parallel to V_2 and $\tilde{\pi}_2$ is projection to V_2 parallel to V_1. □

We now define a matrix $\mathcal{C} \in \Lambda_r \mathrm{SU}(2)$:

$$\mathcal{C} := \frac{-ie^{i\theta}}{\sqrt{|T|^2 + 1}} \begin{pmatrix} 1 & T\lambda \\ \bar{T}\lambda^{-1} & -1 \end{pmatrix} \ ,$$

$$\text{where } \ T = \frac{\alpha^{-1}\bar{A}B(1 + \bar{\alpha}^2)}{|A|^2 - \frac{\bar{\alpha}^2}{|\alpha|^2}|B|^2} \ \text{ and } \ \theta = arg\big(|A|^2 - \frac{\bar{\alpha}^2}{|\alpha|^2}|B|^2\big) \ .$$

Theorem 5.8. *Let ϕ be a solution of $d\phi = \phi\xi$ on $\mathcal{R}$ and let $\phi = FB$ be the r-Iwasawa splitting of ϕ. We assume that the monodromy M_ϕ of ϕ is in $\Lambda_r \mathrm{SU}(2)$ and is $\pm id$ at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$. We do the dressing $\phi \to h \cdot \phi$, then $h\phi = (hF\tilde{h}^{-1}\mathcal{C}^{-1})(\mathcal{C}\tilde{h}B)$ is r-Iwasawa splitting of $h \cdot \phi$, i.e. $hF\tilde{h}^{-1}\mathcal{C}^{-1} \in \Lambda_r \mathrm{SU}(2)$ and $\mathcal{C}\tilde{h}B \in \Lambda_{r+} \mathrm{SL}(2, \boldsymbol{C})$, where $h, \tilde{h}, \mathcal{C}$ are defined as above.*

Proof. We first show $hF\tilde{h}^{-1} \in \Lambda_r SU(2)$. F, $\mathcal{C}$ have the reality condition, i.e $A(\bar{\lambda}^{-1}) = (A(\lambda)^{-1})^*$. We show that $h, \tilde{h}$ also have the reality condition.

$$\begin{aligned}
h(\bar{\lambda}^{-1}) &= f(\bar{\lambda}^{-1})^{-1/2}\pi_1 + f(\bar{\lambda}^{-1})^{1/2}\pi_2 \ , \\
(h(\lambda)^{-1})^* &= (f(\lambda)^{1/2}\pi_1 + f(\lambda)^{-1/2}\pi_2)^* \\
&= \overline{f}(\lambda)^{1/2}\pi_1^* + \overline{f}(\lambda)^{-1/2}\pi_2^* \\
&= f(\bar{\lambda}^{-1})^{-1/2}\pi_1 + f(\bar{\lambda}^{-1})^{1/2}\pi_2 \quad . \\
\tilde{h}(\bar{\lambda}^{-1}) &= f(\bar{\lambda}^{-1})^{-1/2}\tilde{\pi}_1(\bar{\lambda}^{-1}) + f(\bar{\lambda}^{-1})^{1/2}\tilde{\pi}_2(\bar{\lambda}^{-1}) \ , \\
(\tilde{h}(\lambda)^{-1})^* &= f(\lambda)^{1/2}\tilde{\pi}_1(\lambda) + f(\lambda)^{-1/2}\tilde{\pi}_2(\lambda))^* \\
&= \overline{f}(\lambda)^{1/2}\tilde{\pi}_1(\lambda)^* + \overline{f}(\lambda)^{-1/2}\tilde{\pi}_2(\lambda)^* \\
&= f(\bar{\lambda}^{-1})^{-1/2}\tilde{\pi}_1(\bar{\lambda}^{-1}) + f(\bar{\lambda}^{-1})^{1/2}\tilde{\pi}_2(\bar{\lambda}^{-1}) \quad .
\end{aligned}$$

Thus we have shown the reality condition for h, $\tilde{h}$. F, $\mathcal{C}$ are holomorphic on $r < |\lambda| < r^{-1}$. h, $\tilde{h}$ are holomorphic on $r < |\lambda| < r^{-1}$ with singularities only at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$. Thus we need only check that $hF\tilde{h}^{-1}$ has no singularities at $\lambda = \pm\alpha, \pm\bar{\alpha}^{-1}$.

$$\begin{aligned}
hF\tilde{h}^{-1}|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} &= \left((f^{-1/2}\pi_1 + f^{1/2}\pi_2)F(f^{1/2}\tilde{\pi}_1 + f^{-1/2}\tilde{\pi}_2)\right)\big|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} \\
&= (\pi_1 F\tilde{\pi}_1 + f\pi_2 F\tilde{\pi}_1 + f^{-1}\pi_1 F\tilde{\pi}_2 + \pi_2 F\tilde{\pi}_2)|_{\lambda=\pm\alpha,\pm\bar{\alpha}^{-1}} \\
&= C_1 + 0 + 0 + C_2 \qquad (\exists C_1, \exists C_2 \in \mathrm{SL}(2,\boldsymbol{C})) \\
&\neq \infty \quad .
\end{aligned}$$

Finally we show $\mathcal{C}\tilde{h}B \in \Lambda_{+r}SL(2,\boldsymbol{C})$. B is in $\Lambda_{+r}SL(2,\boldsymbol{C})$, so we need only check that $\mathcal{C}\tilde{h}$ is in $\Lambda_{+r}SL(2,\boldsymbol{C})$. We can clearly see $\mathcal{C}\tilde{h} \in \Lambda_r SL(2,\boldsymbol{C})$ and is holomorphic on $0 < |\lambda| < r$ and continuous on $0 < |\lambda| \le r$. A direct computation shows that

$$\mathcal{C}\tilde{h}B|_{\lambda=0} = \begin{pmatrix} \rho_1\rho_0 & 0 \\ 0 & {\rho_1}^{-1}\rho_0^{-1} \end{pmatrix} \ ,$$

where $\rho_1 = \sqrt{\dfrac{|\alpha|^{-1}|A|^2 + |\alpha||B|^2}{|\alpha||A|^2 + |\alpha|^{-1}|B|^2}} \in \boldsymbol{R}_{\rangle 0}$ and $B|_{\lambda=0} = \begin{pmatrix} \rho_0 & 0 \\ 0 & \rho_0^{-1} \end{pmatrix}$.

□

Theorem 5.8 has the following corollary:

Corollary 5.9. *We have explicit parametrizations for cylinder bubbletons in all three space forms using the Sym-Bobenko formulas* (2.13), (2.15) *and* (2.16).

Remark. Cylinder multibubbletons also have explicit parametrizations.

Remark. Proposition 3.2 implies that the conformal factor of the metric of the bubbleton surface is

$$2e^{2u} = 2\rho^2 \ , \tag{5.1}$$

where $B \in \Lambda SL_{r,+}(2,\boldsymbol{C})$ and $B|_{\lambda=} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$.

5.3. **Equivalence of the simple type dressing and Bianchi's Bäcklund transformation on the cylinder.** In this section we prove the equivalence of the simple type dressing and Bianchi's Bäcklund transformation in R^3 in the case of the cylinder. Bianchi's Bäcklund transformation is described in [24]. Actually, in the cylinder case, we can show that the metric, the Hopf differential and mean curvature of Bianchi's Bäcklund trasformation are the same as those resulting from the simple type dressing. In a general setting, Fran Burstall [5] has proven that equivalence of the simple type dressing and Darboux transformation of CMC surfaces. This implies the equivalence of the simple type dressing and Bianchi's Bäcklund transformation, because Udo Hertrich-Jeromin and Franz Pedit [7] have proven that the equivalence of Darboux transformation of CMC surfaces and Bianchi's Bäcklund transformation of CMC surfaces. Thus what we are proving here is only a special case of something that has been recently proven by Fran Burstall. But we include a proof here, because our proof is more direct and tailored to the case for which we need it.

First we introduce the metric, the Hopf differential and the mean curvature of Bianchi's Bäcklund transformation using [24]. Using the notation in [24], we can write the first and second fundamental forms and the principal curvatures of a CMC surface as follows:

$$\begin{cases} ds^2 & = e^{2w}(dx^2+dy^2) \\ II & = e^{w}(\sinh(w)dx^2+\cosh(w)dy^2) \\ k_1 & = e^{-w}\sinh(w), \quad k_2 = e^{-w}\cosh(w) \end{cases}$$

We can compute H and Q:

$$H = 1/2, \quad Q = -1/2 \quad .$$

The Gauss equation becomes as follows:

$$2w_{z\bar{z}} + \sinh(2w) = 0 \quad .$$

In particular, in the cylinder case we have $w = 0$. We do the Bäcklund transformation on the cylinder, and we get the following function w_1 of the new metric $e^{2w_1}(dx^2+dy^2)$:

$$w_1 = 2\tanh^{-1}\Big(\tanh(\beta_1)\frac{\cos(b)}{\cosh(a)}\Big) \quad ,$$

where $a = x\sinh(\beta_1), b = y\cosh(\beta_1)$ and $\beta_1 \in \boldsymbol{C}$. Using $\tanh(z) = 1/2\log(\frac{1+z}{1-z})$, we can rewrite w_1 as follows:

$$e^{2w_1} = \Big(\frac{\cosh(\beta_1)\cosh(a)+\sinh(\beta_1)\cos(b)}{\cosh(\beta_1)\cosh(a)-\sinh(\beta_1)\cos(b)}\Big)^2 \quad .$$

Under Bianchi's Bäcklund transformation, the Hopf differential and mean cuvature do not change.

Next we compute the metric after the dressing h. Using the proof of Theorem 5.8 and equation (5.1), we have the following metric:

$$2e^{2u_1} = 2\rho^2 = 2\rho_1^2\cdot\rho_0^2 = 2\Big(\frac{|\alpha|^{-1}|A|^2+|\alpha||B|^2}{|\alpha||A|^2+|\alpha|^{-1}|B|^2}\Big)^2\cdot\rho_0^2 \quad .$$

In particular, for the cylinder case, $\rho_0 = 1$, $A = \cosh(\frac{\alpha^{-1}z-\alpha\bar{z}}{4})$, $B = \sinh(\frac{\alpha^{-1}z-\alpha\bar{z}}{4})$ and $\alpha \in \boldsymbol{R}$. Note that in this case, we make the change of the coordinate z to e^z for the cylinder example in Section 4.1.

Theorem 5.10. *Bianchi's Bäcklund transformation of the cylinder and the simple type dressing of the cylinder are the same surface.*

Proof. Bianchi's Bäcklund transformation of the cylinder has the following metric, mean curvature and the Hopf differential:

$$e^{2w_1} = \Big(\frac{\cosh(\beta_1)\cosh(a) + \sinh(\beta_1)\cos(b)}{\cosh(\beta_1)\cosh(a) - \sinh(\beta_1)\cos(b)}\Big)^2 \ ,$$
$$H = 1/2 \ ,$$
$$Q = -1/2 \ ,$$

where $a = x\sinh(\beta_1), b = y\cosh(\beta_1)$ and $\beta_1 \in \boldsymbol{C}$.

The simple type dressing by h has the following metric, mean curvature and the Hopf differential:

$$2e^{2u_1} = 2\Big(\frac{\alpha^{-1}|\cosh(X)|^2 + \alpha|\sinh(X)|^2}{\alpha|\cosh(X)|^2 + \alpha^{-1}|\sinh(X)|^2}\Big)^2 \ ,$$
$$H = 1/2 \ ,$$
$$Q = -1 \ ,$$

where $X = \frac{\alpha^{-1}z - \alpha\bar{z}}{2}$.

We consider the change of the coordinate $w = \sqrt{2}\cdot z$. This changes the Hopf differential Q to $(1/2)Q$ and changes the metric $2e^{2u}$ to e^{2u}. Note that the mean curvature doesn't change. Using the additional theorem for hyperbolic sine and cosine functions, we can rewrite the metric:

$$e^{2u_1} = \Big(\frac{(\alpha^{-1}+\alpha)\cosh(X+\bar{X}) - (\alpha^{-1}-\alpha)\cosh(X-\bar{X})}{(\alpha^{-1}+\alpha)\cosh(X+\bar{X}) - (\alpha^{-1}-\alpha)\cosh(X-\bar{X})}\Big)^2 \quad .$$

We have $X + \bar{X} = 2ReX = x(\frac{\alpha^{-1}-\alpha}{2})$ and $X - \bar{X} = 2iImX = iy(\frac{\alpha^{-1}+\alpha}{2})$. Thus the metric has the following form:

$$e^{2u_1} = \Big(\frac{\frac{\alpha^{-1}+\alpha}{2}\cosh x(\frac{\alpha^{-1}-\alpha}{2}) - \frac{\alpha^{-1}-\alpha}{2}\cosh iy(\frac{\alpha^{-1}+\alpha}{2})}{\frac{\alpha+\alpha^{-1}}{2}\cosh x(\frac{\alpha^{-1}-\alpha}{2}) - \frac{\alpha^{-1}-\alpha}{2}\cosh iy(\frac{\alpha^{-1}+\alpha}{2})}\Big)^2 \quad .$$

We put $\frac{\alpha^{-1}+\alpha}{2} = \cosh(\beta_1)$ and $\frac{\alpha^{-1}-\alpha}{2} = \sinh(\beta_1)$. Thus we can rewrite the metric as follows:

$$e^{2u_1} = \Big(\frac{\cosh\beta_1\cosh(x\sinh\beta_1) - \sinh\beta_1\cos(y\cosh\beta_1)}{\cosh\beta_1\cosh(x\sinh\beta_1) - \sinh\beta_1\cos(y\cosh\beta_1)}\Big)^2 \quad .$$

Thus both transformations give the same metric, mean curvature and the Hopf differential. So the fundamental theorem of surface theory implies that the two transformations of the cylinder are the same. □

5.4. **Parallel surfaces of the bubbletons.** CMC surfaces have parallel CMC surfaces. In this section, we prove that the parallel surfaces of the bubbletons are the same surface as the original bubbletons. First we derive a result on parallel CMC surfaces that can be found in [2]:

Theorem 5.11. *Let f be a confromal CMC surface defined by the Sym-Bobenko formula (2.13) on a simply-connected domain $D \subseteq \boldsymbol{R}^2$. Then*

$$f^* = \left[\frac{1}{2}F\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}F^{-1} - i\lambda(\partial_\lambda F)\cdot F^{-1}\right]\Bigg|_{\lambda=1} \quad .$$

is a conformal parametrization of another CMC surface defined for $(x, y) \in D$. We denote the metric, the mean curvature and the Hopf differential of f^* by $2e^{2u^*}(dx^2 + dy^2)$, H^* and Q^*, respectively. Then the conformal factor of $2e^{2u^*}(dx^2+dy^2)$, H^* and Q^* have the following forms:

$$2e^{2u^*} = 2e^{-2u}|Q|^2 \quad ,$$
$$H^* = H \quad ,$$
$$Q^* = Q \quad .$$

Here $2e^{2u}(dx^2 + dy^2)$, H and Q are the metric, the mean curvature and the Hopf differential of the CMC surface f, respectively. We call f^* the parallel surface *of f.*

Proof.

$$(5.2) \qquad (f^*_\lambda)_z = ie^{-u}F\begin{pmatrix} 0 & 0 \\ \lambda^{-1}Q & 0 \end{pmatrix}F^{-1} \ , \quad (f^*_\lambda)_{\bar z} = ie^{-u}F\begin{pmatrix} 0 & \lambda\bar Q \\ 0 & 0 \end{pmatrix}F^{-1} \ .$$

This implies that $\langle f^*_z, f^*_{\bar z}\rangle = 2e^{-2u}|Q|^2$. We can also compute H^* and Q^*.

$$H^* = \frac{1}{2e^{2u^*}}\langle f^*_{z\bar z}, N^*\rangle = H \quad ,$$
$$Q^* = \langle f^*_{zz}, N^*\rangle = Q \quad .$$

Here N and $N^* = -N$ are the normal vectors of the CMC surface and the parallel CMC surface, respectively. □

Theorem 5.12. *The parallel surface of a cylinder bubbleton is the same surface as the original cylinder bubbleton, up to a rigid motion.*

Proof. Using equation (5.1), we can describe the conformal factor of the metric, the mean curvature and the Hopf differential of the cylinder bubbletons as follows:

$$2e^{2u_1} = 2\Big(\frac{\alpha^{-1}|A|^2 + \alpha|B|^2}{\alpha|A|^2 + \alpha^{-1}|B|^2}\Big)^2 \quad ,$$
$$H = 1/2 \quad ,$$
$$Q = -1 \quad ,$$

where $A = \cosh(\frac{\alpha^{-1}z - \alpha\bar z}{4})$, $B = \sinh(\frac{\alpha^{-1}z-\alpha\bar z}{4})$ and $\alpha \in \boldsymbol{R}$.

Using Theorem 5.11, we can also describe the conformal factor of the metric, the mean curvature and the Hopf differential of the bubbleton parallel surface as follows:

$$2e^{2u_1{}^*} = 2e^{-2u} = 2\Big(\frac{\alpha|A|^2 + \alpha^{-1}|B|^2}{\alpha^{-1}|A|^2 + \alpha|B|^2}\Big)^2 \quad ,$$
$$H^* = 1/2 \quad ,$$
$$Q^* = -1 \quad .$$

We consider the conformal change of the coodinate $y \to y - \pi/2$ on the parallel surface, where $z := x + iy$. Under this change, the mean curvature and the Hopf differential do not change. For the metric, $|A|^2$ and $|B|^2$ change to $|B|^2$ and $|A|^2$, respectively. Thus the conformal factor of the metric changes as follows:

$$2e^{2u_1{}^*} = 2\Big(\frac{\alpha|A|^2 + \alpha^{-1}|B|^2}{\alpha^{-1}|A|^2 + \alpha|B|^2}\Big)^2 \longrightarrow 2\Big(\frac{\alpha^{-1}|A|^2 + \alpha|B|^2}{\alpha|A|^2 + \alpha^{-1}|B|^2}\Big)^2 = 2e^{u_1} \quad .$$

Thus both surfaces have the same metric, mean curvature and Hopf differential up to this change of coordinate. Thus the fundamental theorem of surface theory implies the two surfaces are the same. □

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