Self-Parallel Constant Mean Curvature Surfaces

Kenta TOYAMA

January 27, 2004

Abstract

It is a classical result that surfaces made from a constant mean curvature (CMC) H surface by moving in the normal direction the distances $1/2H$ and $1/H$, respectively, are of constant Gaussian curvature $K = 4H^2$ and of constant mean curvature $-H$, respectively. We call them parallel surfaces. In this paper, we study CMC surfaces whose parallel CMC surfaces are congruent to the original surface. In particular, we show that Delaunay surfaces (unduloids and nodoids), the simplest Wente tori, and the simplest bubbletons are all surfaces of this type.

Contents

1 Basic surface theory

All parametrizations $P : U \subset \boldsymbol{R}^2 \to S \subset \boldsymbol{R}^3$ are assumed to be compatible with the oriented normal vector n of the surface S; that is, in $P(U)$,

$$n = \frac{P_u \times P_v}{|P_u \times P_v|}.$$

Let $P(u,\, v)$ be a parametrization at a point $P \in S$ of a surface S, and let $\gamma(s) = P(u(s),\, v(s))$ be a parametrized curve on S, which $\gamma(0) = P$. To simplify the notation, we shall make the convention that all functions to appear below denote their values at the point P. The tangent vector to $\gamma(s)$ at p is $\gamma' = P_u u' + P_v v'$ and

$$dn(\gamma') = n'(u(s),\, v(s)) = \frac{d}{ds} n(u(s),\, v(s)) = n_u u' + n_v v'.$$

Since n_u and n_v belong to $T_p(S)$, we may write

$$\begin{aligned} n_u &= AP_u + BP_v, \\ n_v &= CP_u + DP_v, \end{aligned} \tag{1.1}$$

and therefore,

$$dn(\gamma') = (Au' + Bv')P_u + (Cu' + Dv')P_v;$$

thus, in matrix form with respect to the basis $\{P_u,\, P_v\}$ of $T_P(S)$,

$$dn \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

This shows that in the basis $\{P_u,\, P_v\}$, dn is given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Notice that in general this matrix is not necessarily symmetric.

The expression of the second fundamental form in the basis $\{P_u,\, P_v\}$ is given by

$$\begin{aligned} II_p(\gamma') &= -\langle dn(\gamma'),\, \gamma' \rangle = -\langle n_u u' + n_v v',\, P_u u' + P_v v' \rangle \\ &= L(u')^2 + 2Mu'v' + N(v')^2, \end{aligned}$$

where, since $\langle n,\, P_u \rangle = \langle n,\, P_v \rangle = 0$,

$$\begin{aligned} L &= -\langle n_u,\, P_u \rangle = \langle n,\, P_{uu} \rangle, \\ M &= -\langle n_v,\, P_u \rangle = \langle n,\, P_{uv} \rangle = -\langle n_u,\, P_v \rangle, \\ N &= -\langle n_v,\, P_v \rangle = \langle n,\, P_{vv} \rangle. \end{aligned} \tag{1.2}$$

We shall now compute the values of A, B, C, D in terms of the coefficients L, M, N. From (1.2), we have

$$\begin{aligned} -M &= \langle n_u,\ P_v\rangle = AF + CG, \\ -M &= \langle n_v,\ P_u\rangle = BE + DF, \\ -L &= \langle n_u,\ P_u\rangle = AE + CF, \\ -N &= \langle n_v,\ P_v\rangle = BF + DG, \end{aligned} \tag{1.3}$$

where

$$E = \langle P_u,\ P_u\rangle, \quad F = \langle P_u,\ P_v\rangle, \quad G = \langle P_v,\ P_v\rangle,$$

are the coefficients of the first fundamental form in the basis $\{P_u,\ P_v\}$. Relation (1.3) may be expressed in matrix form by

$$-\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}; \tag{1.4}$$

thus,

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}. \tag{1.5}$$

From (1.5), we get

$$\begin{aligned} A &= \frac{FM - GL}{EG - F^2}, \quad B = \frac{FL - EM}{EG - F^2}, \\ C &= \frac{FN - GM}{EG - F^2}, \quad D = \frac{FM - EN}{EG - F^2}. \end{aligned} \tag{1.6}$$

The relations (1.1), with the above values, are known as the equation of Weingarten. Let $-k_1$, $-k_2$ be the eigenvalues of dn, hence k_1 and k_2 satisfy the equation

$$dn(v_j) = -k_j v = -k_j I v_j \quad \text{for some } v_j \in T_P(s),\ v_j \neq 0,\ j = 1, 2,$$

where I is the identity map. From (1.4), we immediately get

$$K = k_1 k_2 = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{LN - M^2}{EG - F^2}. \tag{1.7}$$

We compute the mean curvature as follows: The linear map $dn + k_j I$ is not invertible, hence it has zero determinant. Thus

$$\begin{aligned} & \det \begin{pmatrix} A + k_j & B \\ C & D + k_j \end{pmatrix} = 0 \\ \iff\quad & {k_j}^2 + (A + D)k_j + AD - BC = 0 \\ \iff\quad & (EG - F^2){k_j}^2 - (EN + GL - 2FM)k_j + LN - M^2 = 0. \end{aligned} \tag{1.8}$$

Since k_1 and k_2 are the roots of the above quadratic, we conclude that

$$H = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}(A + D) = \frac{EN + GL - 2FM}{2(EG - F^2)}. \tag{1.9}$$

From (1.8), we get

$$\det \begin{pmatrix} L - k_j E & M - k_j F \\ M - k_j F & N - k_j G \end{pmatrix} = 0.$$

This is the equation to find k_j such that there exists the solution $w_j = (\xi_j,\ \eta_j)^t \neq (0,\ 0)^t$ of the following equation:

$$\begin{aligned} (L - k_j E)\xi_j + (M - k_j F)\eta_j &= 0, \\ (M - k_j F)\xi_j + (N - k_j G)\eta_j &= 0. \end{aligned} \tag{1.11}$$

Here, k_j are the principal curvatures and $w_j = (\xi_j,\ \eta_j)^t \in T_P(S)$ are the principal directions.

2 Parallel surfaces

2.1 Three parallel surfaces

In this subsection, we show that there exist surfaces of constant Gaussian and constant mean curvature, respectively, parallel to a constant mean curvature nonminimal surface in the normal direction.

Definition 1. *If a regular connected curve C in S is such that for all $P \in C$ the tangent line of C is a principal direction of S at P, then C is said to be a curvature line of S.*

Definition 2. *Let S be an orientable surface and let n be a unit normal vector of S. We consider a surface $\bar{S}$ to be parallel to S if there is a normal geodesic congruence between S and $\bar{S}$ such that the distance between corresponding points is constant, i.e. for each $P \in S$ we have*

$$\bar{P}(u,\ v) = P(u,\ v) + a \cdot n(u,\ v), \tag{2.1}$$

where $a \neq 0$ is a real constant. We say that S and $\bar{S}$ are parallel surfaces at distance a.

Lemma 2.1. *Let S be an orientable surface and let $\bar{S}$ be parallel to S at distance a. Then the unit normal vector $\bar{n}$ of $\bar{S}$ is equal to the unit normal vector n of S.*

Proof. From (2.1), we get

$$\bar{P}_u = P_u + a \cdot n_u = (1 + aA)P_u + BP_v, \tag{2.2}$$

$$\bar{P}_v = P_v + a \cdot n_v = CP_u + (1 + aD)P_v. \tag{2.3}$$

From (2.2) and (2.3), we get $\langle \bar{P}_u,\, n\rangle = 0$, $\langle \bar{P}_v,\, n\rangle = 0$. $\square$

Lemma 2.2. *Two principal directions of $P(u,\, v)$ are orthogonal, away from umbilic points (where $k_1 = k_2$).*

Proof. Let k_i be the principal curvatures and let $w_j = (\xi_j,\, \eta_j)^t$ be the corresponding principal directions. From (1.11), we have

$$\begin{aligned} L\xi_j + M\eta_j &= k_j(E\xi_j + F\eta_j), \\ M\xi_j + N\eta_j &= k_j(F\xi_j + G\eta_j), \end{aligned} \qquad (j = 1,\, 2). \tag{2.4}$$

The equation (2.4) may be expressed in matrix form by

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix} = k_j \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}, \quad (j = 1,\, 2). \tag{2.5}$$

Using (2.5), we get

$$\begin{aligned} k_2 \cdot (\xi_1,\, \eta_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} &= (\xi_1,\, \eta_1) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} \\ &= (\xi_2,\, \eta_2) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \\ &= k_1 \cdot (\xi_2,\, \eta_2) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \\ &= k_1 \cdot (\xi_1,\, \eta_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} \end{aligned}$$

From the assumption $k_1 \neq k_2$, we get

$$(\xi_1,\, \eta_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = 0 \iff \langle w_1,\, w_2\rangle = 0.$$

$\square$

Lemma 2.3. *Let $P(u,\, v)$ be a surface with no umbilic points. Then the parametric curves $P(u,\, v_0)$ and $P(u_0,\, v)$ are curvature lines for each fixed $u_0,\, v_0$ if and only if*

$$F = M = 0. \tag{2.7}$$

Proof. To prove one direction, assume P_u and P_v are principal curvature directions. Then, from Lemma 2.2, we immediately have $F = 0$. Substituting $F = 0$ into (1.11), we get

$$\begin{aligned}(L - k_j E)\xi_j + M\eta_j &= 0, \\ M\xi_j + (N - k_j G)\eta_j &= 0.\end{aligned}$$

Since the principal directions of the curvature lines are the directions of P_u and P_v (so $\eta_1 = 0$ and $\xi_2 = 0$), we have from the above equations that the principal curvatures satisfy

$$k_1 = \frac{L}{E}, \quad k_2 = \frac{N}{G} \tag{2.8}$$

and $M = 0$.

Conversely, if $F = M = 0$, then k_1 and k_2 are as is (2.8) and $\eta_1 = 0$ and $\xi_2 = 0$, hence P_u and P_v are principal directions. □

Remark . Let $P(u, v_0)$ be a principal curve, then substituting (2.7) into (1.1) and (1.6), we get the following equations:

$$n_u = -k_1 P_u, \quad n_v = -k_2 P_v, \tag{2.9}$$

where k_1 and k_2 are the principal curvatures.

Lemma 2.4. *If $P(u, v_0)$ is a principal curve of S, then $\bar{P}(u, v_0)$ is a principal curve of $\bar{S}$.*

Proof. Since $P(u, v_0)$ is a principal curve of S,

$$\bar{P}_u = P_u + a \cdot n_u = (1 - ak_1)P_u. \tag{2.10}$$

Hence

$$P_u = \frac{1}{1 - ak_1}\bar{P}_u. \tag{2.11}$$

From Lemma 2.1 and (2.9), (2.11), we get

$$\bar{n}_u = n_u = -k_1 P_u = -\frac{k_1}{1 - ak_1}\bar{P}_u.$$

With

$$\bar{k}_1 = \frac{k_1}{1 - ak_1}, \tag{2.12}$$

we have $\bar{n}_u = -\bar{k}_1 \bar{P}_u$. This means that $\bar{P}(u, v_0)$ is also a principal curve. Here, $\bar{k}_1$ is the principal curvature of $\bar{S}$ along $\bar{P}(u, v_0)$. □

Let $\tau(u)$ be the torsion of $P(u, v_0)$. Then we have

$$\tau(u) = \frac{\det(P_u, P_{uu}, P_{uuu})}{|P_u \times P_{uu}|^2}. \tag{2.13}$$

Using (2.13), we can show this lemma:

Lemma 2.5. *Let $P(u, v_0)$ be a curve with nonzero curvature in S. Then $P(u, v_0)$ is planar if and only if the torsion of $P(u, v_0)$ is equal to zero, i.e. if and only if we have*

$$\det(P_u,\, P_{uu},\, P_{uuu}) = 0. \tag{2.14}$$

Lemma 2.6. *If $P(u, v_0)$ is a planar principal curve in S, then $\bar{P}(u, v_0)$ is a planar principal curve in $\bar{S}$.*

Proof. Using (2.10) and (2.11), we get

$$\begin{aligned}
\bar{P}_{uu} &= -a(k_1)_u P_u + (1 - ak_1)P_{uu}, \\
\bar{P}_{uuu} &= -a(k_1)_u P_{uu} - a(k_1)_{uu} P_u + (1 - ak_1)P_{uuu} - a(k_1)_u P_{uu} \\
&= -a(k_1)_{uu} P_u - 2a(k_1)_u P_{uu} + (1 - ak_1)P_{uuu}.
\end{aligned}$$

Using (2.14), we get

$$\det(\bar{P}_u,\, \bar{P}_{uu},\, \bar{P}_{uuu}) = (1 - ak_1)^3 \cdot \det(P_u,\, P_{uu},\, P_{uuu}) = 0.$$

□

Proposition 2.7. *Let S be a regular orientable surface and let a be a real constant such that $1 - 2aH + a^2K \neq 0$. Then the curvatures $\bar{H}$ and $\bar{K}$ of the surface $\bar{S}$ parallel to S at a distance a are given by*

$$\bar{K} = \frac{K}{1 - 2aH + a^2K}, \tag{2.15}$$

$$\bar{H} = \frac{H - aK}{1 - 2aH + a^2K}. \tag{2.16}$$

Proof. From (2.12) in Lemma 2.4, we find the principal curvatures of $\bar{S}$:

$$\bar{k}_1 = \frac{k_1}{1 - ak_1}, \qquad \bar{k}_2 = \frac{k_2}{1 - ak_2}. \tag{2.17}$$

Using (1.7), (1.9) and (2.17), we calculate the Gaussian curvature and mean curvature:

$$\begin{aligned}
\bar{K} &= \bar{k}_1 \bar{k}_2 = \frac{k_1}{1 - ak_1} \cdot \frac{k_2}{1 - ak_2} = \frac{k_1 k_2}{1 - 2a \cdot \frac{k_1 + k_2}{2} + a^2 k_1 k_2} \\
&= \frac{K}{1 - 2aH + a^2K}, \\
\bar{H} &= \frac{1}{2}(\bar{k}_1 + \bar{k}_2) = \frac{1}{2}\left(\frac{k_1}{1 - ak_1} + \frac{k_2}{1 - ak_2}\right) = \frac{\frac{k_1 + k_2}{2} - ak_1 k_2}{1 - 2a \cdot \frac{k_1 + k_2}{2} + a^2 k_1 k_2} \\
&= \frac{H - aK}{1 - 2aH + a^2K}.
\end{aligned}$$

□

Corollary 2.8. *Let S be a regular orientable surface with no umbilic points and such that its Gaussian curvature does not vanish.*

If S has constant mean curvature $H > 0$, then there exist two surfaces parallel to S such that one has constant positive Gaussian curvature $K = 4H^2$ and the other one has constant mean curvature equal to $-H$.

If S has positive constant Gaussian curvature K, then there exist two surfaces parallel to S at the distance $a = \mp\dfrac{1}{\sqrt{K}}$ whose mean curvatures are constant and equal to $H = \pm\dfrac{\sqrt{K}}{2}$.

Proof. Suppose S has constant mean curvature $H > 0$. Substituting $a = \dfrac{1}{2H}$ into (2.15) and (2.16), we get

$$\begin{aligned}
\bar{K} &= \frac{K}{1 - 2\cdot\frac{1}{2H}\cdot H + \frac{1}{4H^2}\cdot K} = 4H^2, \\
\bar{H} &= \frac{H - \frac{1}{2H}\cdot K}{1 - 2\cdot\frac{1}{2H}\cdot H + \frac{1}{4H^2}\cdot K} = \frac{4H^3 - 2H^2K}{K}.
\end{aligned}$$

By assumption, we have $K \neq 0$. So the parallel surface at distance $\dfrac{1}{2H}$ has constant Gaussian curvature $4H^2$.

Substituting $a = \dfrac{1}{H}$ into (2.15) and (2.16), we get

$$\begin{aligned}
\bar{H} &= \frac{H - \frac{1}{H}\cdot K}{1 - 2\cdot\frac{1}{H}\cdot H + \frac{1}{H^2}\cdot K} = -H, \\
\bar{K} &= \frac{K}{1 - 2\cdot\frac{1}{H}\cdot H + \frac{1}{H^2}\cdot K} = \frac{H^2K}{H^2 - K}.
\end{aligned}$$

We have

$$H^2 - K = 0 \iff \frac{(k_1 + k_2)^2}{4} - k_1k_2 = \frac{(k_1 - k_2)^2}{4} = 0 \iff k_1 = k_2.$$

By assumption, S has no umbilic points, so $H^2 - K \neq 0$. So the parallel surface at distance $\dfrac{1}{H}$ has constant mean curvature $-H$. The rest of the corollary can be proven with similar arguments. $\square$

Definition 3. *In Corollary 2.8, If S has constant mean curvature $H > 0$, then the constant positive Gaussian curvature surface parallel to S at the distance $a = \dfrac{1}{2H}$ is called the parallel K-surface of S, and the constant mean curvature surface parallel to S at the distance $\dfrac{1}{H}$ is called the parallel H-surface of S.*

Remark . Thus from $P(u,\, v)$, we have three parallel surfaces

$$P(u,\, v),\ P_K(u,\, v),\ P_H(u,\, v),$$

where

$$P_K(u,\, v) = P(u,\, v) + \frac{1}{2H} n, \quad P_H(u,\, v) = P(u,\, v) + \frac{1}{H} n,$$

where n is the unit normal vector of $P(u, v)$. The surface $P_K(u, v)$ represents the parallel K-surface of $P(u,\, v)$ and the surface $P_H(u,\, v)$ represents the parallel H-surface of $P(u,\, v)$.

2.2 Isothermal coordinate systems

When the coordinate system $(u,\, v)$ is isothermal, the coefficients of the first and second fundamental forms are

$$E = G = \lambda, \quad F = 0,$$

and

$$\begin{aligned} L &= \langle P_{uu},\, n\rangle = -\langle P_u,\, n_u\rangle, \\ N &= \langle P_{vv},\, n\rangle = -\langle P_v,\, n_v\rangle, \\ M &= \langle P_{uv},\, n\rangle = -\langle P_u,\, n_v\rangle = -\langle P_v,\, n_u\rangle. \end{aligned}$$

Let us now assume that $(u,\, v)$ are isothermal coordinates for $P(u,\, v)$. Now, we calculate the coefficients of the fundamental forms of the parallel surface $\bar{P}(u,\, v) = P(u,\, v) + a \cdot n$ of the surface $P(u,\, v)$. Using

$$\begin{aligned} n_u &= \frac{FM - GL}{EG - F^2} P_u + \frac{FL - EM}{EG - F^2} P_v = -\frac{L}{\lambda} P_u - \frac{M}{\lambda} P_v, \\ n_v &= \frac{FN - GM}{EG - F^2} P_u + \frac{FM - EN}{EG - F^2} P_v = -\frac{M}{\lambda} P_u - \frac{N}{\lambda} P_v, \end{aligned}$$

we get

$$
\begin{aligned}
\bar{E} &= \langle \bar{P}_u,\ \bar{P}_u \rangle = \langle P_u,\ P_u \rangle + 2a\langle P_u,\ n_u \rangle + a^2 \langle n_u,\ n_u \rangle \\
&= \lambda - 2aL + \frac{a^2(L^2+M^2)}{\lambda}, \\
\bar{F} &= \langle \bar{P}_u,\ \bar{P}_v \rangle = \langle P_u,\ P_v \rangle + 2a\langle P_u,\ n_v \rangle + a^2 \langle n_u,\ n_v \rangle \\
&= -2aM + \frac{a^2 M(L+N)}{\lambda}, \\
\bar{G} &= \langle \bar{P}_v,\ \bar{P}_v \rangle = \langle P_v,\ P_v \rangle + 2a\langle P_v,\ n_v \rangle + a^2 \langle n_v,\ n_v \rangle \\
&= \lambda - 2aN + \frac{a^2(M^2+N^2)}{\lambda}, \\
\bar{L} &= -\langle n_u,\ \bar{P}_u \rangle = -\langle n_u,\ P_u \rangle - a\langle n_u,\ n_u \rangle \\
&= L - \frac{a(L^2+M^2)}{\lambda}, \\
\bar{M} &= -\langle n_v,\ \bar{P}_u \rangle = -\langle n_v,\ P_u \rangle - a\langle n_v,\ n_u \rangle \\
&= M - \frac{aM(L+N)}{\lambda}, \\
\bar{N} &= -\langle n_v,\ \bar{P}_v \rangle = -\langle n_v,\ P_v \rangle - a\langle n_v,\ n_v \rangle \\
&= N - \frac{a(M^2+N^2)}{\lambda}.
\end{aligned}
$$

From Corollary 2.8, we can calculate the coefficients of the fundamental forms of the parallel surfaces when $a = \frac{1}{2H},\ \frac{1}{H}\ \left(H = \frac{L+N}{2\lambda}\right)$.

(i) When $a = \frac{1}{2H} = \frac{\lambda}{L+N}$,

$$
\begin{aligned}
\bar{E}_K &= \lambda - \frac{2\lambda L}{L+N} + \frac{\lambda^2}{(L+N)^2} \cdot \frac{L^2+M^2}{\lambda} = \frac{\lambda(M^2+N^2)}{(L+N)^2}, \\
\bar{F}_K &= -\frac{2\lambda M}{L+N} + \frac{\lambda^2}{(L+N)^2} \cdot \frac{M(L+N)}{\lambda} = -\frac{\lambda M}{L+N}, \\
\bar{G}_K &= \lambda - \frac{2\lambda N}{L+N} + \frac{\lambda^2}{(L+N)^2} \cdot \frac{M^2+N^2}{\lambda} = \frac{\lambda(L^2+M^2)}{(L+N)^2},
\end{aligned}
$$

This shows that the coordinates $(u,\ v)$ are not generally isothermal coordinates of $P_K(u,\ v)$.

(ii) When $a = \dfrac{1}{H} = \dfrac{2\lambda}{L+N}$,

$$\begin{aligned}
\bar{E}_H &= \lambda - \frac{2\cdot 2\lambda L}{L+N} + \frac{4\lambda^2}{(L+N)^2}\cdot\frac{L^2+M^2}{\lambda} \\
&= \frac{\lambda\{(N-L)^2+4M^2\}}{(L+N)^2}, && (2.20)\\
\bar{F}_H &= -\frac{2\cdot 2\lambda M}{L+N} + \frac{4\lambda^2}{(L+N)^2}\cdot\frac{M(L+N)}{\lambda} = 0, \\
\bar{G}_H &= \lambda - \frac{2\cdot 2\lambda N}{L+N} + \frac{4\lambda^2}{(L+N)^2}\cdot\frac{M^2+N^2}{\lambda} \\
&= \frac{\lambda\{(N-L)^2+4M^2\}}{(L+N)^2}, && (2.21)\\
\bar{L}_H &= L - \frac{2\lambda}{L+N}\cdot\frac{L^2+M^2}{\lambda} = \frac{LN-L^2-2M^2}{L+N}, && (2.22)\\
\bar{M}_H &= M - \frac{2\lambda}{L+N}\cdot\frac{M(L+N)}{\lambda} = -M, && (2.23)\\
\bar{N}_H &= N - \frac{2\lambda}{L+N}\cdot\frac{M^2+N^2}{\lambda} = \frac{LN-N^2-2M^2}{L+N}. && (2.24)
\end{aligned}$$

This shows that (u, v) is an isothermal coordinate system of the parallel H-surface $P_H(u,\ v)$ as well.

3 Delaunay surfaces

3.1 Parallel H-surface of Delaunay surfaces

The locus of a focus of an ellipse as the point of contact rolls along a straight line in a plane will be called the undulary. The locus of a focus of a hyperbola as the point of contact rolls along a straight line in a plane forms the curve which we shall call the nodary. Rotating each of the roulettes about its axis of rolling produces five types of surfaces with constant mean curvature in Euclidean three space $\boldsymbol{R}^3$, called Delaunay surfaces: the catenoids (by rolling a parabola), unduloids, nodoids, right circular cylinders (which are unduloids made by rolling a circle), and spheres (by rolling a degenerate ellipse of eccentricity 0). Delaunay surfaces are represented in terms of elliptic integrals, and they are periodic.

In this subsection, we show what parallel H-surface of Delaunay surfaces are and find the relation between the original surface and its parallel H-surface.

Proposition 3.1. *Let H be a positive constant. The mean curvature of the surface of revolution of the planar curve $C : y = y(x)$, $(y > 0)$ around*

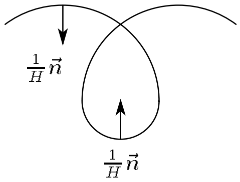

Figure 1: Normal vector along a nodoidal profile curve.

the axis of x is equal to the constant H if and only if y satisfies one of the following differential equations:

$$y^2 - \frac{1}{H} \cdot \frac{y}{\sqrt{1+y'^2}} = 0, \tag{3.1}$$

$$y^2 - \frac{1}{H} \cdot \frac{y}{\sqrt{1+y'^2}} + b^2 = 0, \quad b = \frac{1}{2H}, \tag{3.2}$$

$$y^2 - \frac{1}{H} \cdot \frac{y}{\sqrt{1+y'^2}} + b^2 = 0, \quad \frac{1}{2H} > b > 0, \tag{3.3}$$

$$y^2 - \frac{1}{H} \cdot \frac{y}{\sqrt{1+y'^2}} - b^2 = 0, \tag{3.4}$$

where b is a constant. The solution of (3.1) represents a half circle. The solution of (3.2) represents part of a straight line. The solution of (3.3) represents part of an undulary, and the solution of (3.4) represents part of a nodary.

Proof. See [4], [5] and [8]. □

Now we think about the parallel H-surfaces of Delaunay surfaces. Clearly, the H-surface of a Delaunay surface is again a Delaunay surface.

Theorem 3.2. *The parallel H-surface of an unduloid, resp. nodoid, is congruent to the original unduloid, resp. nodoid.*

Proof. From (3.3) and (3.4), we have

$$y^2 - \frac{1}{H} \cdot \frac{y}{\sqrt{1+y'^2}} \pm b^2 = 0.$$

We calculate the values of y such that $y' = 0$, i.e. the extremal values. See Figure 1. Substituting $y' = 0$ into (3.3), we get

$$y^2 - \frac{y}{H} \pm b^2 = 0 \iff y = \frac{1 \pm \sqrt{1 \mp 4H^2b^2}}{2H}.$$

At these extremal points, the values of y at the corresponding points on the parallel H-surface are

$$\frac{1 \pm \sqrt{1 \mp 4H^2b^2}}{2H} - \frac{1}{H} = \frac{-1 \pm \sqrt{1 \mp 4H^2b^2}}{2H}.$$

Since the H-surface also has extremal values at these points, its profile curve equation at $y' = 0$ becomes a quadratic equation with solutions $\dfrac{-1 \pm \sqrt{1 \mp 4H^2b^2}}{2H}$:

$$y^2 + \frac{y}{H} \pm b^2 = 0.$$

Since $b < \dfrac{1}{2H}$, we find

$$y = \frac{-1 \pm \sqrt{1 - 4H^2b^2}}{2H} < 0.$$

i.e. the value of y for the H-surface of an unduloid is negative. Now, since the Delaunay H-surface revolves around the same axis as the original Delaunay surface, we can change y to $-y$ and get the same surface, and then

$$y^2 - \frac{y}{H} \pm b^2 = 0.$$

This is the equation

$$y^2 - \frac{1}{H} \cdot \frac{y}{\sqrt{1 + y'^2}} \pm b^2 = 0$$

with $y' = 0$ substituted in. Therefore, we find that the H-surfaces of unduloids and nodoids are congruent to the original surfaces, and the values of b are also the same. $\square$

Remark . The simplest example of Proposition 3.2 occurs with the right circular cylinders, which are a special case of the unduloids. The right circular cylinder is parametrized by u and v, i.e.

$$P(u,\ v) = (r\cos u,\ r\sin u,\ v), \tag{3.6}$$

where r is a constant. Using (3.6), we calculate a unit normal vector n and mean curvature H:

$$\begin{aligned} n &= \frac{P_u \times P_v}{|P_u \times P_v|} = (\cos u,\ \sin u,\ 0), && (3.7) \\ H &= \frac{-r}{2r^2} = -\frac{1}{2r}. && (3.8) \end{aligned}$$

Using (3.7) and (3.8), we can calculate the parameter of H-surface of the right circular cylinder:

$$\begin{aligned} \bar{P}(u,\ v) &= P(u,\ v) + \frac{1}{H} \cdot n \\ &= (r\cos u,\ r\sin u,\ v) - 2r(\cos u,\ \sin u,\ 0) \\ &= (-r\cos u,\ -r\sin u,\ v). \end{aligned}$$

Thus, we find that the parallel H-surface of the right circular cylinder is congruent to the original surface.

Remark . The simplest example of constant mean curvature surfaces for which there do not exist parallel surfaces occurs with spheres. The sphere is parametrized by u and v, i.e.

$$P(u,\ v) = (r \cos u \cos v,\ r \cos u \sin v,\ r \sin u), \tag{3.9}$$

where r is a constant. Using (3.9), we calculate the unit normal vector n, mean curvature H and Gaussian curvature K:

$$n = \frac{P_u \times P_v}{|P_u \times P_v|} = (-\cos u \cos v,\ -\cos u \sin v,\ -\sin u), \tag{3.10}$$

$$H = \frac{r^3 \cos^2 u + r^3 \cos^2 u}{2r^4 \cos^2 u} = \frac{1}{r}, \tag{3.11}$$

$$K = \frac{r^2 \cos^2 u}{r^4 \cos^2 u} = \frac{1}{r^2}. \tag{3.12}$$

Using (3.11) and (3.12), we get $H^2 - K = 0$. This means that all points in the sphere are umbilic points. Using (3.10), (3.11) and (3.12), we calculate the coordinate of H-surface of the sphere:

$$\bar{P}(u,\ v) = P(u,\ v) + \frac{1}{H} \cdot n = (0,\ 0,\ 0).$$

Since the H-surface is just one point, there does not exist a parallel H-surface for the sphere.

3.2 Undularies and unduloids

In this and next subsection, we derive explicit formulas for unduloids and nodoids, and their parallel H-surface and K-surface. We use these explicit formulas to make the computer graphics of these surfaces in this EG-Model.

Now the ellipse in the left-hand side of Figure 2 corresponds with the other in the right-hand side. We calculate the distances between the points in the right figure. Now we assume $a > b > 0$. From the right figure, we find

$$\begin{aligned} PF_1^2 &= (a\cos\theta + \sqrt{a^2 - b^2})^2 + b^2 \sin^2\theta \\ &= (a^2 - b^2)\left(\cos^2\theta + \frac{2a\cos\theta}{\sqrt{a^2 - b^2}} + \frac{a^2}{a^2 - b^2}\right) \\ &= (a^2 - b^2)\left(\cos\theta + \frac{a}{\sqrt{a^2 - b^2}}\right)^2. \end{aligned}$$

From the assumption, since $a > b$ and $\dfrac{a}{\sqrt{a^2 - b^2}} > 1$, we find

$$\cos\theta + \frac{a}{\sqrt{a^2 - b^2}} > 0$$

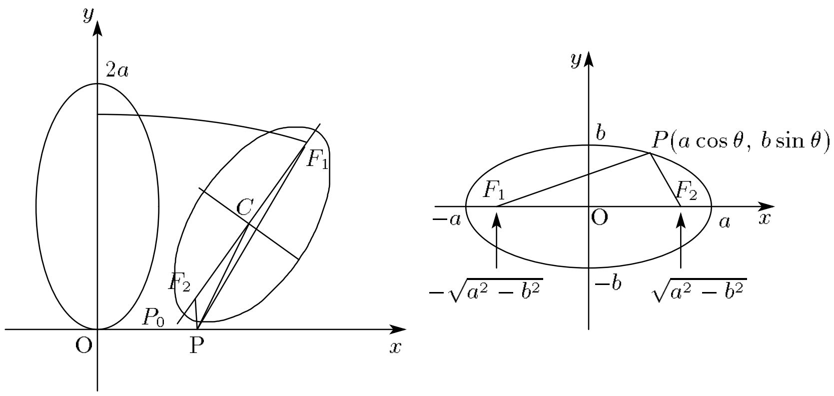

Figure 2: Rolling of an ellipse.

and get

$$PF_1 = \left(\cos\theta + \frac{a}{\sqrt{a^2 - b^2}}\right)\sqrt{a^2 - b^2} = a + \sqrt{a^2 - b^2}\cdot\cos\theta. \qquad (3.14)$$

Similarly,

$$PF_2 = a - \sqrt{a^2 - b^2}\cdot\cos\theta. \qquad (3.15)$$

Let Q_1 be the foot of a perpendicular from F_1 to the axis of x and let Q_2 be the foot of a perpendicular from F_2 to the axis of x. Now we calculate the distances F_1Q_1 and F_2Q_2:

$$\begin{aligned}
{F_1Q_1}^2 &= \frac{\left|\dfrac{b\cos\theta}{a\sin\theta}\sqrt{a^2 - b^2} - \dfrac{b}{\sin\theta}\right|^2}{\left(\sqrt{\dfrac{b^2\cos^2\theta}{a^2\sin^2\theta} + 1}\right)^2} \\
&= \frac{b^2(a^2 - b^2)\cos^2\theta + a^2b^2 + 2ab^2\sqrt{a^2 - b^2}\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}, && (3.16) \\
{F_2Q_2}^2 &= \frac{\left|-\dfrac{b\cos\theta}{a\sin\theta}\sqrt{a^2 - b^2} - \dfrac{b}{\sin\theta}\right|^2}{\left(\sqrt{\dfrac{b^2\cos^2\theta}{a^2\sin^2\theta} + 1}\right)^2} \\
&= \frac{b^2(a^2 - b^2)\cos^2\theta + a^2b^2 - 2ab^2\sqrt{a^2 - b^2}\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}. && (3.17)
\end{aligned}$$

Using (3.14)–(3.17), we get

$$\begin{aligned}
\sin^2 \angle F_1PQ_1 &= \frac{\overline{F_1Q_1}^2}{\overline{PF_1}^2} = \frac{b^2}{a^2\sin^2\theta + b^2\cos^2\theta}, \\
\sin^2 \angle F_2PQ_2 &= \frac{\overline{F_2Q_2}^2}{\overline{PF_2}^2} = \frac{b^2}{a^2\sin^2\theta + b^2\cos^2\theta}.
\end{aligned}$$

Since $\angle F_1PQ_1$ and $\angle F_2PQ_2$ are acute angles, we get

$$\angle F_1PQ_1 = \angle F_2PQ_2 := \alpha.$$

We immediately get

$$\begin{aligned}
&\sin^2\alpha = \frac{b^2}{a^2\sin^2\theta + b^2\cos^2\theta}, \\
&\cos^2\alpha = 1 - \sin^2\alpha = \frac{(a^2-b^2)\sin^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}, \\
\Longleftrightarrow \quad &\begin{cases} \sin\alpha = \dfrac{b}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} > 0, \\[2ex] \cos\alpha = \dfrac{\sqrt{a^2-b^2}\cdot\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}. \end{cases}
\end{aligned} \tag{3.19}$$

And, since the distance of $P(\theta) = OP$ is equal to an arc P_0P of the ellipse, we get

$$OP = P(\theta) = \int_0^\theta \sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}\ d\varphi \tag{3.20}$$

Using (3.14), (3.19) and (3.20), F_1 is parametrized as φ, i.e.

$$\begin{aligned}
x &= OP + PF_1\cdot\cos\alpha \\
&= \int_0^\theta \sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}\ d\varphi + \frac{(a+\sqrt{a^2-b^2}\cdot\cos\theta)\sqrt{a^2-b^2}\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}},
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
y &= PF_1\cdot\sin\alpha \\
&= \frac{b(a+\sqrt{a^2-b^2}\cdot\cos\theta)}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.
\end{aligned} \tag{3.22}$$

Using (3.21) and (3.22), we can draw three parallel surfaces of an undulary and unduloid, using Mathematica. See the file delaunay.nb in this EG-Model.

3.3 Nodaries and nodoids

Now we think about the curve $\tilde{C}$ of the solution parametrized as an arc length s:

$$x = x(s), \quad y = y(s).$$

Now since

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

as φ is a function of class C^1 of a parameter s, we can change

$$\frac{dx}{ds} = \sin\varphi(s), \quad \frac{dy}{ds} = \cos\varphi(s).$$

Since we find

$$1 + f'^2 = 1 + \left(\frac{dy}{dx}\right)^2 = \frac{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}{\left(\frac{dx}{ds}\right)^2} = \frac{1}{\sin^2\varphi}, \tag{3.23}$$

substituting (3.23) into (3.4), we get

$$y^2 - \frac{|\sin\varphi|}{H} y - b^2 = 0.$$

Changing $\dfrac{1}{H} = 2A$, we get

$$y^2 - 2A|\sin\varphi| y - b^2 = 0 \iff y = A|\sin\varphi| \pm \sqrt{A^2\sin^2\varphi + b^2}.$$

Changing $b^2 = A^2\beta$, we get

$$y = A|\sin\varphi| \pm |A|\sqrt{\sin^2\varphi + \beta}, \quad \beta > 0.$$

From the assumption of $y > 0$, we have to think about two cases:

$$\begin{aligned} &\text{when} \quad A > 0, \quad y = A(|\sin\varphi| + \sqrt{\sin^2\varphi + \beta}), \\ &\text{when} \quad A < 0, \quad y = -A(-|\sin\varphi| + \sqrt{\sin^2\varphi + \beta}). \end{aligned}$$

So, assuming the following:

$$\begin{aligned} &\text{when } A > 0, \ \frac{dx}{ds} = \sin\varphi \geq 0, \\ &\text{when } A < 0, \ \frac{dx}{ds} = \sin\varphi \leq 0, \end{aligned}$$

in spite of a sign of A, we get

$$y = |A|(\sin\varphi + \sqrt{\sin^2\varphi + \beta}).$$

Changing $|A| = a$, we get

$$\begin{aligned} y &= a(\sin\varphi + \sqrt{\sin^2\varphi + \beta}), \qquad (3.24)\\ \beta &= \frac{b^2}{a^2}. \end{aligned}$$

From (3.23), since we have

$$\frac{dx}{dy} = \tan\varphi,$$

we get

$$\begin{aligned} x &= \int \tan\varphi \cdot \frac{dy}{ds} ds = y\tan\varphi - \int y \frac{d\tan\varphi}{ds} ds \\ &= y\tan\varphi - \int \frac{y}{\cos^2\varphi} d\varphi, \\ x &= x_0 + y\tan\varphi - \int_0^\varphi \frac{y}{\cos^2\varphi} d\varphi, \qquad (3.25)\\ x_0 &= x|_{\varphi=0}. \end{aligned}$$

Substituting (3.24) into (3.25), we get

$$x = x_0 + a(\sin\varphi + \sqrt{\sin^2\varphi + \beta})\tan\varphi - a\int_0^\varphi \frac{\sin\varphi + \sqrt{\sin^2\varphi + \beta}}{\cos^2\varphi} d\varphi.$$

By the way, as $\omega = \cos\varphi$, since we have

$$\begin{aligned} \int_0^\varphi \frac{\sin\varphi}{\cos^2\varphi} d\varphi &= \int_1^{\cos\varphi} \frac{-1}{\omega^2} d\omega = \left[\frac{1}{\omega}\right]_1^{\cos\varphi} = \frac{1}{\cos\varphi} - 1, \qquad (3.27)\\ \tan\varphi\sin\varphi &= \frac{\sin^2\varphi}{\cos\varphi} = \frac{1}{\cos\varphi} - \cos\varphi, \qquad (3.28)\\ \int_0^\varphi \frac{\sqrt{\sin^2\varphi + \beta}}{\cos^2\varphi} d\varphi &= \left[\tan\varphi \cdot \sqrt{\sin^2\varphi + \beta}\right]_0^\varphi - \int_0^\varphi \tan\varphi \frac{\sin\varphi\cos\varphi}{\sqrt{\sin^2\varphi + \beta}} d\varphi \\ &= \tan\varphi \cdot \sqrt{\sin^2\varphi + \beta} - \int_0^\varphi \frac{\sin^2\varphi}{\sqrt{\sin^2\varphi + \beta}} d\varphi, \qquad (3.29) \end{aligned}$$

from (3.27), (3.28) and (3.29), we get

$$x = x_0 + a\left(a - \cos\varphi + \int_0^\varphi \frac{\sin^2\psi}{\sqrt{\sin^2\psi + \beta}} d\psi\right). \qquad (3.30)$$

Using (3.24) and (3.30), we can draw three parallel surfaces of a nodary and nodoid, using Mathematica. Again see the file delaunay.nb in this EG-Model.

4 Wente tori

In this section, we shall give explicit formulas for the original Wente tori and parallel H-surfaces of Wente tori, based on [19]. We use these explicit formulas to make computer graphics of these surfaces. Later, we shall assume that the mean curvature H is $\frac{1}{2}$, but for now we shall only assume that H is a nonzero constant. Let $P : \boldsymbol{C}/\Gamma \to \boldsymbol{R}^3$ be a conformal immersion of class C^∞ where $\boldsymbol{C}/\Gamma$ is a compact 2-dimensional torus determined by the 2-dimensional lattice Γ. Note that $(u,\, v)$ are then isothermal coordinates on $\boldsymbol{C}/\Gamma$. The fundamental forms and the Gaussian and mean curvature functions are

$$I = \lambda(du^2 + dv^2), \quad II = L du^2 + 2M du dv + N dv^2,$$
$$K = \frac{LN - M^2}{\lambda^2}, \quad H = \frac{L+N}{2\lambda}.$$

Since H is constant, the Hopf differential $\Phi\, dz^2$ is holomorphic, where $\Phi = \frac{1}{2}(L-N) - iM$ and $z = u + iv$. Thus Φ is constant and P has no umbilics points. Moreover, by a change of the coordinates $(u,\, v)$, we may assume $\Phi = 1$ and so

$$M = 0, \quad L = e^f + 1, \quad N = e^f - 1, \tag{4.1}$$

and $(u,\, v)$ become curvature line parameters, where $f : \boldsymbol{C}/\Gamma \to \boldsymbol{R}$ is defined by

$$HE = e^f \iff H\lambda = e^f. \tag{4.2}$$

We have the equations of Gauss and Weingarten:

$$P_{uu} = \frac{1}{2} f_u P_u - \frac{1}{2} f_v P_v - (e^f + 1) n, \tag{4.3}$$
$$P_{vv} = -\frac{1}{2} f_u P_u + \frac{1}{2} f_v P_v - (e^f - 1) n, \tag{4.4}$$
$$P_{uv} = \frac{1}{2} f_v P_u + \frac{1}{2} f_u P_v, \tag{4.5}$$

$$n_u = H(1 + e^{-f}) P_u, \quad n_v = H(1 - e^{-f}) P_v, \tag{4.6}$$
$$\Delta f + 4H \sinh f = 0, \tag{4.7}$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ and $n : \boldsymbol{C}/\Gamma \to \boldsymbol{R}^3$ is the unit normal vector field, i.e. the Gauss map. Therefore the problem of finding constant mean curvature immersed tori in $\boldsymbol{R}^3$ reduces to solving the PDE system (4.3)–(4.7) by real analytic functions f, n, P defined on $\boldsymbol{R}^2$ and doubly periodic with respect to some fundamental lattice $\Gamma \subset \boldsymbol{R}^2$. In the case of the original Wente tori, in Walter's notation, the solution f of (4.7) is:

$$f = 4 \tanh^{-1}\{\gamma \cdot \bar{\gamma} \cdot cn_k(\alpha u) \cdot cn_{\bar{k}}(\bar{\alpha} v)\}, \tag{4.8}$$

where cn_k denotes the Jacobi amplitudinus cosinus function with modulus k, and $k = \sin\theta$, $\bar{k} = \sin\bar{\theta}$, for $\theta,\ \bar{\theta} \in \left(0,\ \frac{\pi}{2}\right)$ and $\theta + \bar{\theta} < \frac{\pi}{2}$, and

$$\gamma = \sqrt{\tan\theta}, \quad \bar{\gamma} = \sqrt{\tan\bar{\theta}}, \tag{4.9}$$
$$\alpha = \sqrt{4H\frac{\sin 2\bar{\theta}}{\sin 2(\theta+\bar{\theta})}}, \quad \bar{\alpha} = \sqrt{4H\frac{\sin 2\theta}{\sin 2(\theta+\bar{\theta})}}.$$

Now we assume $H = \frac{1}{2}$.

Lemma 4.1 (Walter). *The set of all original Wente tori are in a one-to-one correspondence with the set of reduced fractions* $\frac{\ell}{n} \in (1, 2)$.

For each $\frac{\ell}{n}$, we call the corresponding Wente torus $W_{\ell/n}$. Following Walter's notation, each $W_{\ell/n}$ has either one or two planar geodesic loops in the central symmetry plane: two loops if ℓ is odd, and one loop if ℓ is even. Each loop can be partitioned into $2n$ congruent curve segments, and ℓ is the total winding order of the Gauss map along each loop. The conditions for double periodicity of the position vector function P are expressed in terms of θ and $\bar{\theta}$. Walter determined that there is exactly one

$$\bar{\theta} \cong 65.354955^\circ$$

that solves one period problem, and $\bar{\theta}$ is independent of $\frac{\ell}{n}$. The solution θ in the next lemma solves the other remaining period problem:

Lemma 4.2 (Walter). *For any* $\frac{\ell}{n} \in (1, 2)$, *there is exactly one solution* $\theta \in \left(0,\ \frac{\pi}{2} - \bar{\theta}\right)$ *of*

$$\int_0^{\frac{\pi}{2}} \frac{1 + \tan\theta\tan\bar{\theta}\cos^2\varphi}{1 - \tan\theta\tan\bar{\theta}\cos^2\varphi}\frac{d\varphi}{\sqrt{1 - \sin^2\theta\sin^2\varphi}} = \frac{\ell}{n}\frac{\pi}{2}\sqrt{\frac{\sin 2\bar{\theta}}{\sin 2(\theta+\bar{\theta})}}, \tag{4.11}$$

and for any $\frac{\ell}{n} \notin (1, 2)$ *there is no solution* $\theta \in \left(0,\ \frac{\pi}{2} - \bar{\theta}\right)$ *of (4.11).*

Remark . This is proven in [19], and we can verify this by a nonrigorous numerical computation: setting

$$\mu(\theta) = \sqrt{\sin 2(\theta+\bar{\theta})}\int_0^{\frac{\pi}{2}} \frac{1 + \tan\theta\tan\bar{\theta}\cos^2\varphi}{1 - \tan\theta\tan\bar{\theta}\cos^2\varphi}\frac{d\varphi}{\sqrt{1 - \sin^2\theta\sin^2\varphi}},$$

we find that $\mu(\theta)$ is a monotone function for $0 < \theta < \frac{\pi}{2} - \bar{\theta}$.

Lemma 4.3. *Let $W_{\ell/n}$ have range $[T_{\min}, T_{\max}]$ for K, then no other $W_{\hat{\ell}/\hat{n}}$ with $\hat{\ell}/\hat{n} \neq \ell/n$ has the same range for K.*

Proof. Using (1.7), (4.1) and (4.2), we calculate the Gaussian curvature K:

$$K = \frac{LN - M^2}{EG} = \frac{e^{2f} - 1}{4e^{2f}} = \frac{1 - e^{-2f}}{4}. \tag{4.12}$$

Since

$$-1 \leq cn_k(\alpha u) \leq 1, \quad -1 \leq cn_{\bar{k}}(\bar{\alpha} v) \leq 1,$$

we get

$$-\gamma\bar{\gamma} \leq \gamma\bar{\gamma} \cdot cn_k(\alpha u) \cdot cn_{\bar{k}}(\bar{\alpha} v) \leq \gamma\bar{\gamma}. \tag{4.14}$$

With Γ defined by

$$\frac{\Gamma}{4} = \tanh^{-1} \gamma\bar{\gamma}, \tag{4.15}$$

equation (4.8) and inequality (4.14) imply that

$$-\Gamma \leq f \leq \Gamma. \tag{4.16}$$

Substituting (4.16) into (4.12), we get that the range of K is precisely

$$\left[\frac{1 - e^{-2\Gamma}}{4}, \frac{1 - e^{2\Gamma}}{4}\right]. \tag{4.17}$$

From (4.9), (4.15) and (4.17), we find that the range of K is determined by θ. Therefore, from Lemma 4.1 and Lemma 4.2, no two distinct $W_{\ell/n}$'s can have the same range. □

Using Lemma 4.3, we can find the following fact.

Theorem 4.4. *For all $\dfrac{\ell}{n}$, the parallel H-surface $\overline{W_{\ell/n}}$ of the Wente torus $W_{\ell/n}$ is the same as the original Wente torus, i.e. $\overline{W_{\ell/n}}$ is congruent to $W_{\ell/n}$.*

Proof. The surfaces $W_{\ell/n}$ are the complete collection of constant mean curvature tori foliated by planar principal curves (see [16]). So, from Lemma 2.6, we find that the parallel H-surface $\overline{W_{\ell/n}}$ of the Wente torus $W_{\ell/n}$ is another Wente torus $W_{\hat{\ell}/\hat{n}} = \overline{W_{\ell/n}}$. So we need only calculate the range of the Gaussian curvature $\bar{K}_H$ of the parallel H-surface $\overline{W_{\ell/n}}$ and see that it equals the range of K for $W_{\ell/n}$. Then Lemma 4.3 implies $W_{\ell/n}$ is the same surface as $\overline{W_{\ell/n}}$. By (2.20)–(2.24), we get

$$\begin{aligned} \bar{K}_H &= \frac{\bar{L}_H \bar{N}_H - \bar{M}_H^2}{\bar{E}_H \bar{G}_H} \\ &= \frac{\{(LN - L^2 - 2M^2)(LN - N^2 - 2M^2) - M^2(L+N)^2\}(L+N)^2}{\lambda\{(N-L)^2 + 4M^2\}^2}. \end{aligned} \tag{4.18}$$

Substituting (4.1) and (4.2) into (4.18), we get

$$\bar{K}_H = \frac{1-e^{2f}}{4}.$$

Using (4.16), we get that the range of $\bar{K}_H$ is precisely

$$\left[\frac{1-e^{-2\Gamma}}{4}, \frac{1-e^{2\Gamma}}{4}\right]. \tag{4.19}$$

Comparing (4.17) with (4.19), from Lemma 4.3 we find that the parallel H-surface of the Wente torus is congruent to the original Wente torus $W_{\ell/n}$. □

Proposition 4.5 (Walter). *If one allows two simple, noniterated integrals of one real variable, then $P : \boldsymbol{R}^2 \to \boldsymbol{R}^3$ can be represented explicitly in terms of trigonometric functions. We obtain the following representation $P : \boldsymbol{R}^2 \to \boldsymbol{R}^3$ with (nonisothermic) curvature line parameters:*

$$P = \left(Z\cos(w-j) + \frac{\cos w}{2H},\ Z\sin(w-j) + \frac{\sin w}{2H},\ x_3 \right), \tag{4.20}$$

where

$$\begin{aligned}
Z &= \sqrt{\frac{2}{H}}\frac{1}{\bar{\alpha}^2}\cdot\frac{\{(\bar{\alpha}^2-b)\gamma^2\cos^2 u + p\}\bar{\gamma}\cos v - \{p\gamma^2\cos^2 u + (\bar{\alpha}^2+b)\}\gamma\cos u}{\sqrt{p-2b\gamma^2\cos^2 u - p\gamma^4\cos^4 u}\cdot(1-T\cos u\cos v)}, \\
w &= \sqrt{H}\frac{2}{\alpha}\int_0^u \frac{1+T^2\cos^2 t}{1-T^2\cos^2 t}\frac{dt}{\sqrt{1-k^2\sin^2 t}}, \\
j &= \begin{cases} \tan^{-1}\left(\dfrac{1}{\sqrt{H}}\dfrac{\alpha}{2}\tan u\cdot\sqrt{1-k^2\sin^2 u}\right) + (m-1)\pi, \\ \qquad\qquad \left[\dfrac{(2m-3)\pi}{2} \le u < \dfrac{(2m-1)\pi}{2},\quad m\in \boldsymbol{N}\right], \end{cases} \\
x_3 &= \frac{1}{\sqrt{H}}\frac{1}{\bar{\alpha}}\cdot\left\{2T\frac{\cos u\sin v\sqrt{1-\bar{k}^2\sin^2 v}}{1-T\cos u\cos v} + \frac{1}{\bar{\gamma}}\int_0^v \frac{1-2\bar{k}^2\sin^2 t}{\sqrt{1-\bar{k}^2\sin^2 t}}dt\right\}, \\
T &= \gamma\bar{\gamma}.
\end{aligned}$$

Proof. See [19]. □

Remark . To draw Wente tori $W_{\ell/n}$, we give the relation between $W_{\ell/n}$ and the values of θ, and the ranges of u, v. For the values of θ, see [11]. The range of v is fixed: $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$, and the range of u is determined by the number of fundamental pieces. If g is the number of fundamental pieces, the range of u is $-\frac{\pi}{2} \le u \le \frac{(2g-1)\pi}{2}$.

Using (4.20), we can draw parallel surfaces for three Wente tori. See the file wente.nb in this EG-Model.

$W_{\ell/n}$	θ	g	range of u	range of v
$W_{4/3}$	$12.7898°$	3	$-\pi/2 \leq u \leq 5\pi/2$	$-\pi/2 \leq v \leq \pi/2$
$W_{3/2}$	$17.7324°$	4	$-\pi/2 \leq u \leq 7\pi/2$	$-\pi/2 \leq v \leq \pi/2$
$W_{6/5}$	$8.0983°$	5	$-\pi/2 \leq u \leq 9\pi/2$	$-\pi/2 \leq v \leq \pi/2$

Table 1: values for solution θ (the range of u and v are computed using the value $H = 1/2$).

5 A remark on bubbletons and Smyth surfaces

In this section, we briefly remark on the parallel surfaces of bubbletons and Smyth surfaces. Rigorous definitions of these surfaces can be found in the references.

Remark . The bubbletons with cylinder ends and one set of lobes are self-parallel. (See the software "CMCLab" of N. Schmitt [12], or the paper of Shimpei Kobayashi [15] on bubbletons.) However, bubbletons with Delaunay ends and one set of lobes are not self-parallel, because the set of lobes will be shifted with respect to the bulges and necks of the Delaunay ends on the parallel surface.

Bubbletons with cylinder ends and two sets of lobes are also not self-parallel in general. For example, if one set of lobes has two lobes and the other has three lobes, on the parallel surface one set of lobes will be rotated by 90 degrees and the other will be rotated by 60 degrees. So the parallel H-surface is not congruent to the original surface.

These bubbleton examples have no umbilic points, so they show us that being free of umbilic points does not imply the surface is self-parallel.

Remark . Smyth surfaces are simply-connected constant mean curvature surfaces with an internal rotational symmetry. See [7] and [13]. Smyth surfaces are also clearly not self-parallel, since Smyth surfaces have an umbilic point: From the proof of Corollary 2.8, we know that the parallel H-surface is not immersed with respect to the coordinate frame given by the original Smyth surface.

Acknowledgements

I would like to thank Professor Takeshi Sasaki and Professor Wayne Rossman who took good care of me, and my fellows in the W.I.S.K.Y. seminar (Shimpei Kobayashi, Yuji Morikawa and Isao Nagamine) for their helpful suggestions.

References

[1] U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew Math. **374** 169–192 (1987).

[2] C. Adachi, *An introduction to the differential geometry*, Baifukan (1976). (in Japanese).

[3] A. I. Bobenko, *Surfaces in term of 2 by 2 matrices. Old and new integrable cases*, In: Fordy, A.P., Wood, J.(eds) Harmonic Maps and Integrable Systems, Braunschweig Wiesbaden: Vieweg 83–127 (1994).

[4] C. Delaunay, *Sur la surface de révolution dont la courbure moyenne est constante*, J. Math. Pures et Appl. Ser.1,**6** 309–320 (1841).

[5] J. Eells, *The surfaces of Delaunay*, The Mathematical Intelligencer, Springer-Verlag, **9**, No.1, 53–57 (1987).

[6] K. Kenmotsu, *A course in smooth surfaces theory - An introduction to constant mean curvature surfaces*, Baifukan (2000). (in Japanese).

[7] M. L. P. Kilian, *Constant mean curvature cylinders*, doctoral thesis, University of Massachusetts at Amherst (2000).

[8] M. Koiso, *On the surfaces of Delaunay*, Kyoto University of Education Press. Vol.**97** (2000). (in Japanese).

[9] L. L. de Lima, V. F. de Sousa Neto and W. Rossman, *Lower bounds for index of Wente tori*, Hiroshima Math. J. **31** 183–199 (2001).

[10] Manfredo P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall (1976).

[11] W. Rossman, *The morse index of Wente Tori*, Geometriae Dedicata **86**, 129–151 (2001).

[12] N. Schmitt, CMCLab software, Center for Geometry, Analysis, Numerics and Graphics (GANG), University of Massachusetts at Amherst. `http://www.gang.umass.edu/cmc/`

[13] B. Smyth, *A generalization of a theorem of Delaunay on constant mean curvature surfaces*, IMA Volumes in Mathematics and its Applications, Springer-Verlag, 51:137–160 (1993).

[14] Syoshichi Kobayashi, *Differential geometry of curves and surfaces*, Shokabo (1977). (in Japanese).

[15] Shimpei Kobayashi, *Bubbletons in 3-dimensional space forms via the DPW method*, preprint (2002).

[16] J. Spruck, *The elliptic sinh-Gordon equation and the construction of toroidal soap bubbles*, Lecture Notes in Math. 1340, Springer-Verlag, 275–301 (1988).

[17] K. Tenenblat, *Transformations of manifolds and applications to differential equations* 22–30, LONGMAN (1998).

[18] M. Umehara and K. Yamada, *Curves and surfaces*, to appear (2002). (in Japanese).

[19] R. Walter, *Explicit examples to the H-problem of Heinz Hopf*, Geom. Dedicata **23** 187–213 (1987).

[20] H. Wente, *Counterexample of a conjecture of H. Hopf*, Pacific J. Math **121** 193–243 (1986).