TRIPLY PERIODIC MINIMAL SURFACES OF GENUS 4

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It is known that a compact minimal surface in a 3-dimensional flat torus $T^3 = \mathbb{R}^3/\Lambda$ can be regarded as a triply-periodic minimal surface in $\mathbb{R}^3$.

Let M be a compact Riemann surface and $f : M \to T^3$ a conformal minimal surface. Then f can be represented as follows (the Weierstrass representation):

$$(\star) \qquad f(p) = \mathrm{Re}\int_{p_0}^{p} \Phi \qquad (\mathrm{mod}\ \Lambda), \qquad \Phi = \begin{pmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{pmatrix} \eta,$$

where g is a meromorphic function and η is a holomorphic 1-form on M so that

$$(1+|g|^2)^2|\eta|^2$$

gives a Riemannian metric on M, and

$$\textbf{(P)} \qquad \left\{ \mathrm{Re}\oint_\gamma \Phi \;\middle|\; \gamma \in H_1(M,\mathbb{Z}) \right\} \quad \text{is a sublattice of } \Lambda.$$

For such a minimal surface f, we define f_θ as

$$f_\theta = \mathrm{Re}\int_{p_0}^{p} e^{i\theta}\Phi.$$

Note that f_θ is defined only on the universal cover of M in general, even though $f = f_0$ is well-defined on M itself, in T^3. If f_θ is well-defined on M for some torus $\mathbb{R}^3/\Lambda_\theta$, then it is called an *associate surface* of f. In particular, the associate surface $f_{\pi/2}$, if well-defined in a torus, is called the *conjugate surface* of f. Nagano and Smyth [3] gave a criterion for the existence of associate surfaces. Namely, every element of the one parameter family of isometric minimal surfaces is either an associate surface or else is dense in $\mathbb{R}^3$.

Meeks [2] introduced the following:

Definition 1. $f : M \to T^3$ is said to satisfy *property P* if f_θ is well-defined on M (for some torus) for a countable dense set of angles θ (the tori depending on θ).

In [6], the second author constructed a new example of a minimal surface into a flat torus which has a conjugate surface and satisfies property P, as follows:

Let M be a trigonal Riemann surface of genus 4 defined by $w^3 = z^6 - 1$. (See Figure 1). We set

$$g = z, \qquad \eta = \frac{dz}{w^2}$$

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and Λ is given (defined) by the beta function $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$ as

$$\Lambda=\begin{pmatrix} A & A/2 & 0 \\ 0 & \sqrt{3}A/2 & 0 \\ 0 & 0 & 2^{-2/3}A \end{pmatrix}, \qquad \text{where} \qquad A=\frac{1}{2}B\left(\frac{1}{3},\frac{1}{6}\right)$$

Then f defined as in $(\star)$ is well-defined on M into $\mathbb{R}^3/\Lambda$. Also, f has a conjugate surface and satisfies the property P for lattices Λ_θ.

Note that several examples which satisfy property P and have the linear symmetries of a cube (the expression "linear symmetries of a cube" is precisely defined in [2]) are known. For example, Schwarz surfaces [5] and A. Schoen's surfaces [4]. However, our trigonal minimal surface does not have the linear symmetries of a cube (see Figure 2).

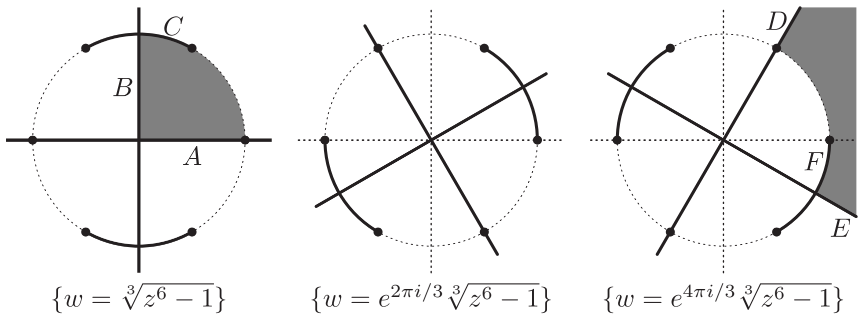

FIGURE 1

M can be represented by the above three Riemann spheres. Thick points are branch points of the branched 3-covering $(z,w)\mapsto z$. Thick lines and curves indicate the planar geodesics, because the Hopf differential is real on these lines and curves (See [1]).

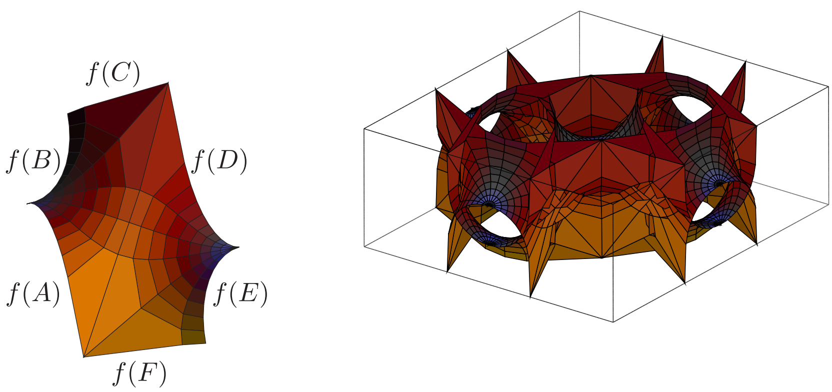

FIGURE 2

Left: A piece $f(D_1 \cup D_2)$ of the fundamental piece of f, which is bounded by six planar geodesics, where $D_1 = \{(z, \sqrt[3]{z^6 - 1}) \in M \,|\, |z| \le 1, 0 \le \arg z \le \pi/2\}$ and $D_2 = \{(z, e^{4\pi i/3}\sqrt[3]{z^6 - 1}) \in M \,|\, |z| \ge 1, -\pi/6 \le \arg z \le \pi/3\}$. See the shaded parts in Figure 1. Right: The fundamental piece of f for the translational symmetry group, which consists of twelve copies of $f(D_1 \cup D_2)$ produced by reflections along planar geodesics.

By the right hand side of Figure 2, we see that f has self-intersections.

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