TRIPLY PERIODIC MINIMAL SURFACES OF GENUS 4

SHOICHI FUJIMORI AND TOSHIHIRO SHODA

It is known that a compact minimal surface in a 3-dimensional flat torus \( T^3 = \mathbb{R}^3 / \Lambda \) can be regarded as a triply-periodic minimal surface in \( \mathbb{R}^3 \).

Let \( M \) be a compact Riemann surface and \( f : M \to T^3 \) a conformal minimal surface. Then \( f \) can be represented as follows (the Weierstrass representation):

\[
(*) \quad f(p) = \text{Re} \int_{p_0}^p \Phi \quad (\text{mod } \Lambda), \quad \Phi = \left( \frac{1 - g^2}{i(1 + g^2)} \right) \eta,
\]

where \( g \) is a meromorphic function and \( \eta \) is a holomorphic 1-form on \( M \) so that

\[
(1 + |g|^2)^2 |\eta|^2
\]
gives a Riemannian metric on \( M \), and

\[
(P) \quad \left\{ \text{Re} \oint_\gamma \Phi \bigg| \gamma \in H_1(M, \mathbb{Z}) \right\} \quad \text{is a sublattice of } \Lambda.
\]

For such a minimal surface \( f \), we define \( f_\theta \) as

\[
f_\theta = \text{Re} \int_{p_0}^p e^{i\theta} \Phi.
\]

Note that \( f_\theta \) is defined only on the universal cover of \( M \) in general, even though \( f = f_0 \) is well-defined on \( M \) itself, in \( T^3 \). If \( f_\theta \) is well-defined on \( M \) for some torus \( \mathbb{R}^3 / \Lambda_\theta \), then it is called an associate surface of \( f \). In particular, the associate surface \( f_{\pi/2} \), if well-defined in a torus, is called the conjugate surface of \( f \). Nagano and Smyth [3] gave a criterion for the existence of associate surfaces. Namely, every element of the one parameter family of isometric minimal surfaces is either an associate surface or else is dense in \( \mathbb{R}^3 \).

Meeks [2] introduced the following:

**Definition 1.** \( f : M \to T^3 \) is said to satisfy property \( P \) if \( f_\theta \) is well-defined on \( M \) (for some torus) for a countable dense set of angles \( \theta \) (the tori depending on \( \theta \)).

In [6], the second author constructed a new example of a minimal surface into a flat torus which has a conjugate surface and satisfies property \( P \), as follows:

Let \( M \) be a trigonal Riemann surface of genus 4 defined by \( w^3 = z^6 - 1 \). (See Figure 1). We set

\[
g = z, \quad \eta = \frac{dz}{w^2}
\]

**Date:** February 16, 2006.

2000 **Mathematics Subject Classification.** 53A10, 53C42.

**Key words and phrases.** minimal surface, triply periodic surface.
and $\Lambda$ is given (defined) by the beta function $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ as

$$
\Lambda = \begin{pmatrix}
A & A/2 & 0 \\
0 & \sqrt{3}A/2 & 0 \\
0 & 0 & 2^{-2/3}A
\end{pmatrix}, \quad \text{where} \quad A = \frac{1}{2} B \left( \frac{1}{3}, \frac{1}{6} \right)
$$

Then $f$ defined as in $(\ast)$ is well-defined on $M$ into $\mathbb{R}^3/\Lambda$. Also, $f$ has a conjugate surface and satisfies the property P for lattices $\Lambda_\theta$.

Note that several examples which satisfy property P and have the linear symmetries of a cube (the expression “linear symmetries of a cube” is precisely defined in [2]) are known. For example, Schwarz surfaces [5] and A. Schoen’s surfaces [4]. However, our trigonal minimal surface does not have the linear symmetries of a cube (see Figure 2).

$M$ can be represented by the above three Riemann spheres. Thick points are branch points of the branched 3-covering $(z,w) \mapsto z$. Thick lines and curves indicate the planar geodesics, because the Hopf differential is real on these lines and curves (See [1]).
Left: A piece \( f(D_1 \cup D_2) \) of the fundamental piece of \( f \), which is bounded by six planar geodesics, where 
\[
D_1 = \{(z, \sqrt[6]{z^6-1}) \in M \mid |z| \leq 1, 0 \leq \arg z \leq \pi/2\} \quad \text{and} \quad D_2 = \{(z, e^{2\pi i/3} \sqrt[6]{z^6-1}) \in M \mid |z| \geq 1, -\pi/6 \leq \arg z \leq \pi/3\}.
\]
See the shaded parts in Figure 1. Right: The fundamental piece of \( f \) for the translational symmetry group, which consists of twelve copies of \( f(D_1 \cup D_2) \) produced by reflections along planar geodesics.

By the right hand side of Figure 2, we see that \( f \) has self-intersections.

References


(Fujimori) Department of Mathematics, Kobe University, Kobe 657-8501 Japan
E-mail address: fujimori@math.kobe-u.ac.jp

(Shoda) Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan
E-mail address: tshoda@math.kyushu-u.ac.jp